Hong Liu lectures on:

[Holographic Duality & Applications]

Lecture Notes by

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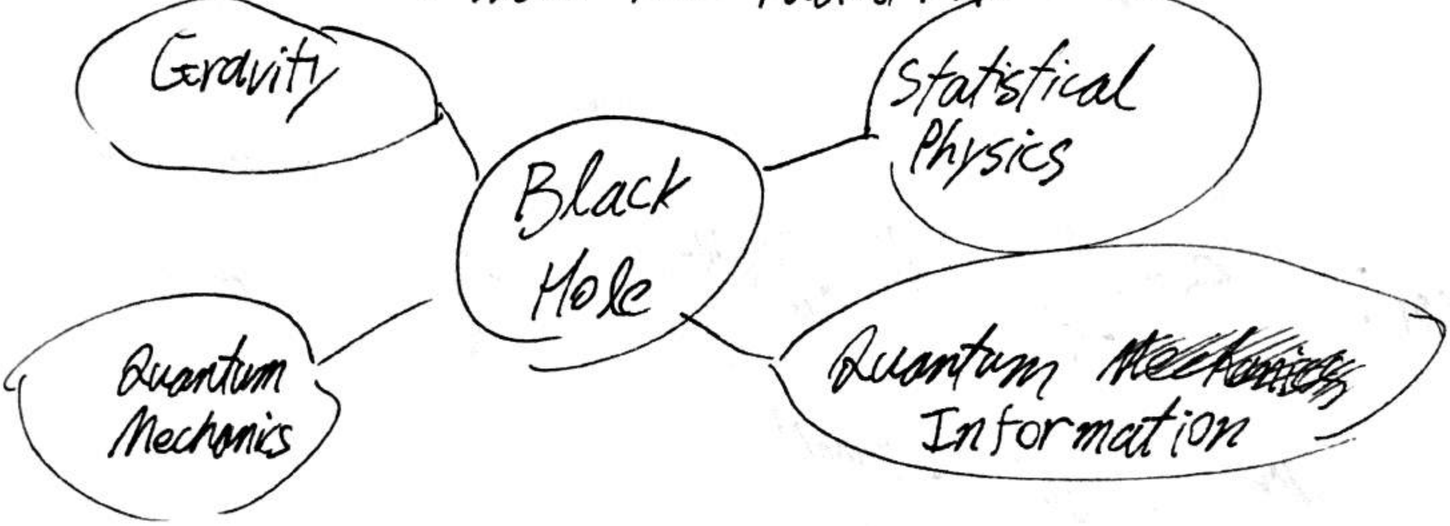
Chapter 1:
Black Holes & the Holographic Principle

BHs: 1) Key object – astrophysically ubiquitous

2) Quantum Matter around BH

→ Hawking's 1975 paper

↝ Black Hole radiation



↝ Black holes bring Quantum Gravity to a Macroscopic level.

1.1 General Remarks on Gravity

all other interactions: probed to 10^{-33} cm (Large Hadron Collider)

gravity: only 10^{-2} cm

General Relativity: "gravity = spacetime"

Quantum Gravity: " = quantum spacetime"

Question: What is ~~the~~ the relationship between quantum gravitational effects and the nature of spacetime?

Answer: A) Einstein Gravity & Gravitons

line element: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$

Einstein's Equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 8\pi G_N T_{\mu\nu}$ (*)

cosmological const.

matter (i.e. stress-energy tensor)

Action: $S = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\,(R - 2\Lambda) + S_{matter}$ (**)

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

take $\Lambda = 0$, $T_{\mu\nu} = 0$

simplest solution to (*) $\rightarrow \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

"Weak gravity" $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ (1) with $\kappa^2 = 8\pi G_N$

putting (1) into (**), we get

$$S = \int d^4 x \left[\mathcal{L}_2 + \kappa \mathcal{L}_3 + \kappa^2 \mathcal{L}_4 + \cdots \right] + \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu} + O(\kappa^2)$$

canonically normalized

starts at $O(\kappa^2) \simeq O(h^2)$ since $\eta_{\mu\nu}$ solves EOM

cancels $16\pi G_N$

EOM from $\mathcal{L}_2$ give plane wave solutions
These are what we call "gravitational waves"

$\mathcal{L}_2$ is quadratic in $h \Rightarrow$ free field theory for h
$\hookrightarrow$ quantize $\mathcal{L}_2 \Rightarrow$ spin-2 massless particle
"graviton"

$\rightsquigarrow \mathcal{L}_3, \mathcal{L}_4, \ldots \rightarrow$ self-interaction of the graviton

gravitational interaction of matter
is by exchange of an $h_{\mu\nu}$
e.g. electron as matter has $T_{\mu\nu} \sim \bar{\psi}\psi$

$\Rightarrow$ 

Treated as QFT, S is non-renormalizable.
$\rightsquigarrow$ So we can treat $S[h]$ as an effective field theory
not fundamental.

B) Important Scales of Gravity

Planck Scales: $\hbar, G_N, c = 1 \Rightarrow$

$$M_p = \sqrt{\frac{\hbar}{G_N}} \simeq 1.2 \cdot 10^{19}\,\text{GeV}$$
$$l_p = \frac{\hbar}{M_p} = \sqrt{\hbar G_N} \simeq 1.6 \cdot 10^{-33}\,\text{cm}$$
$$t_p = l_p \simeq 5.4 \cdot 10^{-44}\,\text{sec}$$

For reference: top quark mass: $\approx 10^{-17} M_p$
electron mass: $\approx 10^{-23} M_p$

Largeness of $M_p \leftrightarrow$ weakness of gravity at microscopic scales
$\Rightarrow$ Consider two particles of mass m brought to nearest possible distance

$\rightsquigarrow \lambda_G := \frac{V_g(r_c)}{m} = \frac{1}{m}\frac{G_N m^2}{r_c} = \frac{G_N m^2}{\hbar} = \frac{m^2}{M_p^2}$ $\qquad r_c = \hbar/m$ "Compton wavelength"

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We see $\lambda_G = \frac{m^2}{M_p^2} = \frac{l_p^2}{r_c^2}$. For $m \ll M_p$, $\lambda_G \ll 1$ e.g. $e^- \to \lambda_G \sim 10$

For $m \sim M_p$, $\lambda_G \sim O(1) \Rightarrow$ Quantum Gravity

Relativistic calculation: $\lambda_G \sim E^2/M_p^2$, E c.o.m. energy

If we take $m \gg M_p$, does $\lambda_G \gg 1$? No!

Different question: take point particle of mass m. At what distance r_s from it does classical gravity become strong?

probe $m' \to \frac{\frac{G m m'}{r_s}}{m'} \sim 1 \Rightarrow r_s = G_N m$

Remarks: 1) In Newtonian gravity, at r_s escape velocity $\sim c$

2) In GR, $r_s \sim$ Schwarzschild radius of Black Hole

$\Rightarrow r_s \sim$ minimal distance we can probe an obj. in classical gravity.

Two important scales: $r_c \sim \hbar/m$, $r_s \sim G_N m$ $\Rightarrow \frac{r_s}{r_c} = \frac{G_N m^2}{\hbar} = \left(\frac{m}{M_p}\right)^2$

1) $m \ll M_p \Rightarrow r_s \ll r_c$ so Compton wavelength outside r_s

$\Rightarrow$ gravitation is weak, negligible [r_s not important quantum effects dominate

2) $m \sim M_p$, $r_s \sim r_c$, $\lambda_G \sim 1$ Quantum Gravity becomes important

3) $m \gg M_p$, $r_s \gg r_c \Rightarrow r_c$ not relevant $\to$ Classical gravity dominates

The relationship between Black Holes and quantum gravity, however, affects much more than Planck scale physics.

Last time.

$$\alpha_G \sim \frac{G_N E^2}{\hbar} \sim \frac{E^2}{M_p^2}$$

$\kappa h_{\mu\nu} T^{\mu\nu}$, $\kappa^2 = 8\pi G_N$ κE κE

$\frac{r_s}{r_c} \sim \frac{m^2}{M_p^2}$

corollary: l_p is minimal localization length

non-grav: $\delta l \sim \hbar/E$

with gravity: $E \sim M_p$

$r_s \sim G_N E \sim r_c \sim l_p$

$E \gg M_p$

$r_s \gg r_c$

$\delta p \sim \frac{\hbar}{\delta\lambda} \Rightarrow \delta\lambda > G\,\delta p \sim \frac{G\hbar}{\delta x}$

$\rightarrow \delta x^2 > \sqrt{G\hbar} \sim l_p$

$E \ll M_p$, ignore: (1) grav. interaction (2) fluctuations of $g_{\mu\nu}$ $\Rightarrow$ QFT in rigid spacetime (can be curved) i.e. on earth.

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Mathematical Treatment:

E fixed, $\hbar$ fixed, $G_N \to 0$

($l_p \to 0$, $M_p \to \infty$)

C low energy expansion in G_N

$$Z = \int Dg\, D\Psi\, e^{iS[g,\Psi]}$$

$$S = \frac{1}{16\pi G_N} S_{grav}[g] + \frac{1}{\lambda} S_m[g,\Psi]$$

λ: matter coupling

$\lambda \gg G_N$

$G_N \to 0 \Rightarrow$ saddle point: $\delta S_{grav}[g] = 0$

$\Rightarrow g_{classical}$

Expand $g = g_{classical} + \kappa h$

$$\Rightarrow S = \frac{1}{16\pi G_N} S_{grav}[g_c] + \underbrace{\frac{1}{\lambda} S_m[\Psi, g_c]}_{\text{QFT in curved spacetime}} + S_2[h] + \cdots + \kappa h_{\mu\nu} T^{\mu\nu}$$

small G_N expansion breaks down at $\frac{E^2}{M_p^2} \sim O(1)$.

For a sphere of radius L, p is quantized as $\frac{1}{L}$

$$\Rightarrow E^2 \sim p^2 \sim \frac{1}{L^2} \sim R$$

6 $\frac{G_N E^2}{\hbar}$

D. gravity in general dimensions

$$S_{grav} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g}\,(R - 2\Lambda)$$

$$[G_d] = \frac{L^{d-1}}{MT^2} \Rightarrow M_{pd}^{d-2} = \frac{\hbar^{d-3}}{G_d}, \quad l_{pd}^{d-2} = \hbar G_d$$

$$\lambda_G \sim \frac{G_d E^{d-2}}{\hbar^{d-3}} \sim \frac{E^{d-2}}{M_{pd}^{d-2}}$$

$$r_s \sim (G_N m)^{\frac{1}{d-3}}$$

consider $M_d = M_D \times Y$

M_D non-compact, D-dimensional

Y compact, $d-D$

suppose Y is too small to be detected.

$R \times S^1$

The effective Newton constant G_D for an observer is not the same as the fundamental

$$\frac{1}{G_D} = \frac{1}{G_d} V_Y \leftarrow \text{volume of } Y$$

$$l_{pD}^{D-2} = \frac{l_{pd}^{d-2}}{L^{d-D}}, \quad \text{expect } L > l_{pd}$$

$$\Rightarrow l_{pD} < l_{pd}$$

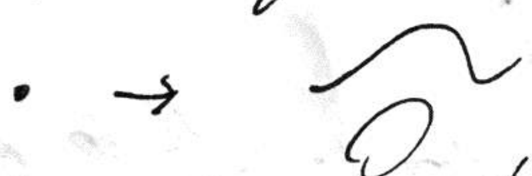
E. Einstein gravity as E.F.T.

gravity tested to 10^{-2} cm

we're going down to 10^{-33} cm

1) Extra dimensions
will see d-dim gravity
before reaching l_p

2) string theory
l_s string length



3) Suppose new physics appears at some scale $l \sim \frac{1}{M}$

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left[R - 2\Lambda + \frac{a_1}{M^2} R^2 + \frac{a_2}{M^2} R_{\mu\nu} R^{\mu\nu} + \dots \right]$$

1.2 | Classical BH Geometry I

consider a spherically symmetric, electrically neutral object of mass M

The Schwarzschild solution (4D) can be analytically found to be:

$$ds^2 = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2\,d\Omega^2$$
$$= g_{\mu\nu}\,dx^\mu dx^\nu$$

with $f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r}$

$r_s := 2G_N M$

Most important features:

1) $r \to \infty$, $f \to 1$, $g_{\mu\nu} \to \eta_{\mu\nu}$

2) $r = r_s$, $f \to 0$, $g_{tt} = 0$
$g_{rr} = \infty$

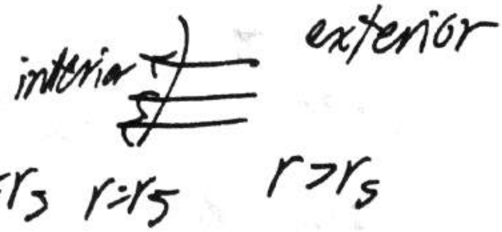
will see t (Schwarzschild time) becomes singular at $r = r_s$

$r = \text{const} > r_s \Rightarrow$ time-like hypersurface

$r = \text{const} < r_s \Rightarrow$ space-like hypersurface

$r = r_s \Rightarrow$ null hypersurface

3) $r = r_s$: event horizon



4) $r = r_s$: hypersurface of infinite redshift

consider an observer $\mathcal{O}_h$ at $r = r_h \gtrsim r_s$

— — — — $\mathcal{O}_\infty$ at $r = \infty$

proper time for $\mathcal{O}_\infty$: t

proper time for $\mathcal{O}_h$

$$d\tau_h = f^{1/2}\, dt = \left(1 - \frac{r_s}{r_h}\right)^{1/2} dt$$

consider a physical process at $r = r_h$ with local proper energy ε

$\mathcal{O}_\infty$ sees energy $E_\infty = \varepsilon \left(1 - \frac{r_s}{r_h}\right)^{1/2}$

as $r_h \to r_s$, $E_\infty \to 0$ "infinite redshift"

$$\boxed{ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega_2^2}$$

$f = 1 - \frac{r_s}{r}$, $\quad r_s = 2GM$

causal structure & Rindler spacetime

consider $r \gtrsim r_s \qquad \frac{r - r_s}{r_s} \ll 1$

proper distance ρ from $r = r_s$

s.t. $d\rho^2 = \frac{dr^2}{f} \Rightarrow d\rho = \frac{dr}{\sqrt{f}}$

$$f(r) = \cancel{f(r_s)} + f'(r_s)(r - r_s) + \cdots$$

$$\frac{dr}{f}$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_s)}}\sqrt{r - r_s}$$

$$\Rightarrow f(r) = \left[\frac{1}{2}f'(r_s)\right]^2 \rho^2 = \kappa^2 \rho^2$$

$$\Rightarrow ds^2 = -\kappa^2 \rho^2 \underline{dt}^2 + d\rho^2 + r_s^2 d\Omega^2$$

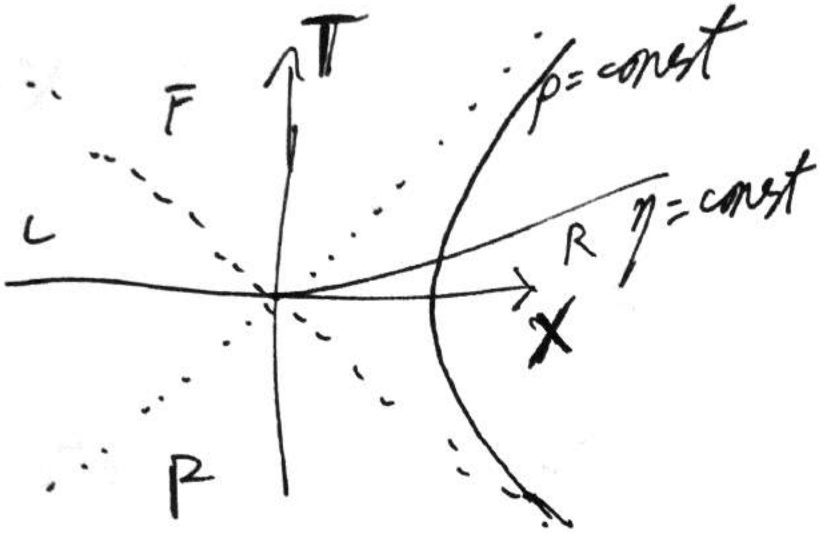
$$= \underbrace{-\rho^2 d\eta^2 + d\rho^2}_{\text{Mink}_2 \text{ (Rindler spacetime)}} + r_s^2 d\Omega^2$$

$\eta = \kappa t$

$$ds^2 = -dT^2 + dX^2 = -\rho^2 d\eta^2 + d\rho^2$$

$$\Updownarrow$$

$$X = \rho\cosh\eta \qquad T = \rho\sinh\eta$$



$$\rho^2 = X^2 - T^2$$
$$\tanh \eta = T/X$$

$X = T = 0 \Rightarrow \rho \to 0,\ \eta$ finite

~~At~~

$X = T \Rightarrow \rho \to 0,\ \eta \to +\infty$ s.t. ρe^{η} finite

$X = -T \Rightarrow \rho \to 0,\ \eta \to -\infty$ s.t. $\rho e^{-\eta}$ finite

Rindler observers: $\rho = \text{const}$ ($\Rightarrow r = \text{const}$)

$\Rightarrow a_{prop} = 1/\rho$

Note| No signal can propogate from F to R

$X = T$: future horizon (can only go in)

$X = -T$: past horizon (can only come out)

1) $r = r_s \leftrightarrow \rho = 0 \leftrightarrow X = \pm T$
null hypersurface

(ρ, η) singular at $\rho = 0 \in (t, r)$ singular at $r = r_s$

2) $r=$ const observer
$\Leftrightarrow$ $\rho=$ const rindler observer

their accelerations agree

3) free-fall observer in BH $\Leftrightarrow$ inertial observer in Rindler
Minkowski

4) Using (T, X), we can extend the black hole geometry from $r \gg r_s$ to <u>four regions</u> with the near-horizon metric

$$ds^2 = -dT^2 + dX^2 + r_s^2 \, d\Omega^2$$

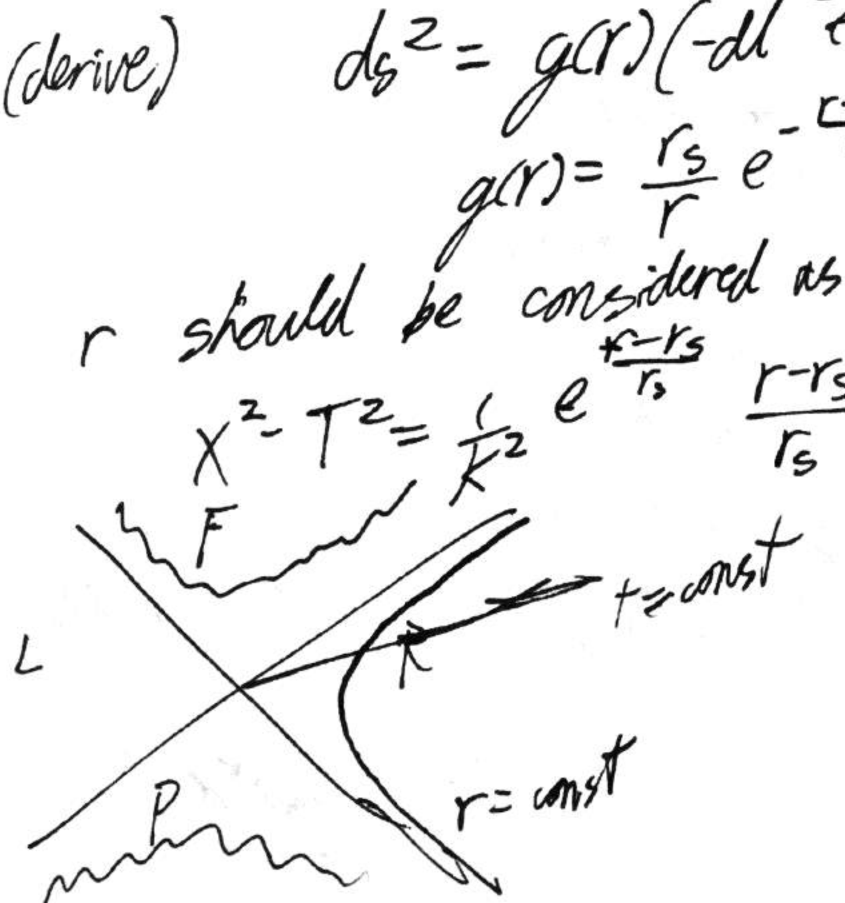
T, X as coordinate transform of (r,t) and then extend them to <u>full spacetime</u>

(derive) $$ds^2 = g(r)(-dT^2 + dX^2) + r_s^2 \, d\Omega_2^2$$

$$g(r) = \frac{r_s}{r} e^{-\frac{r-r_s}{r_s}}$$

r should be considered as a function of (X, T)

$$X^2 - T^2 = \frac{1}{k^2} e^{\frac{r-r_s}{r_s}} \frac{r-r_s}{r_s}$$



a) $g(r_s) = 1$

$X^2 - T^2 = 0$
$(r = r_s)$

b) singularity at $r=0$
$\Leftrightarrow T^2 - X^2 = \frac{1}{k^2} > 0$

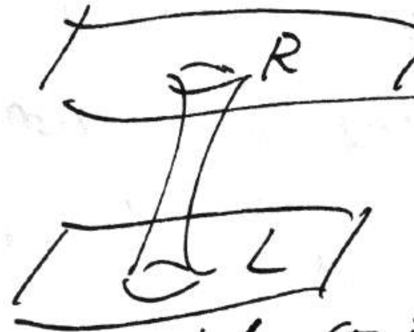
c) symmetries

(i) $T \leftrightarrow -T \qquad X \leftrightarrow -X$

(ii) boost in $(T, X) \Leftrightarrow t \to t + \text{const}$

d) L is a mirror of R w/ another asymptotic flat region

e) $T=0$ slice



wormhole (E-R)
non-traversible

f) F: interior of BH
(future horizon)

g) P: white hole
(past horizon)

h) L, P not present in collapse of a star

A digression: Penrose diagrams

Procedure. 1. choose a metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

x^μ covers full spacetime

2. Find $x^\mu = x^\mu(y^\alpha)$ s.t. y^α has finite range

$$ds^2 = g'_{\alpha\beta}\, dy^\alpha dy^\beta$$

3. construct a new metric

$$\tilde{ds}^2 = \Omega^2(y)\, ds^2 = \tilde{g}_{\alpha\beta}\, dy^\alpha dy^\beta$$

so that the causal structure of $\tilde{g}$ is known

Mink$_2$: $ds^2 = -dT^2 + dX^2$ $\quad U = T - X$

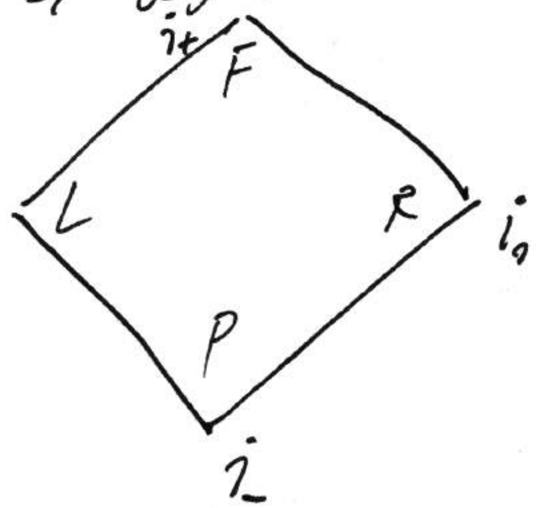
$= -dU\,dV$ $\quad V = T + \cancel{X}$

$U = \tan u \Rightarrow u, v \in [-\pi/2, \pi/2]$

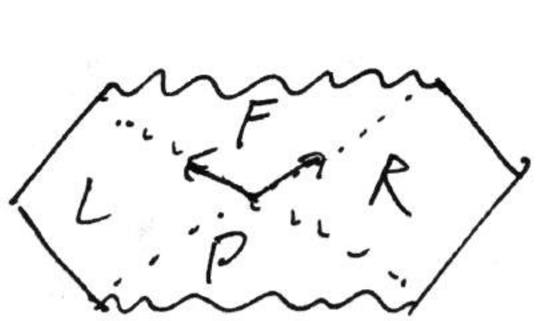
$V = \tan v \quad ds^2 = -\frac{1}{\cos^2 u \cos^2 v}\, du\, dv$

$\Rightarrow \widetilde{ds}^2 = -du\,dv$

$\Rightarrow$

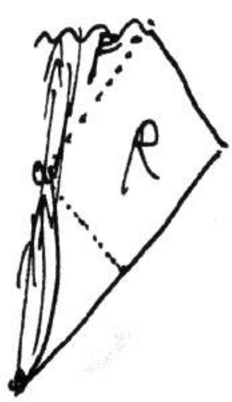


Black Hole:



Kruskal coordinates (X, T)

steller callapse:



Formulas we will use going forward

$$ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2$$

$$= -\kappa^2\rho^2 dt^2 + d\rho^2 + r_s^2\, d\Omega^2 + \ldots \leftarrow \text{near horizon}$$

$$\kappa = \frac{1}{2} f'(r_s) = \frac{1}{2r_s} = \frac{1}{4G_N M}$$

1.3 Black Hole temperature

1975: Hawking
1976: Unruh
Bisognano - Wichmann

Both phenomena at level of leading order in low-energy approx
QFT in a rigid curved spacetime
This effect is universal insofar as it would apply to any QFT regardless of interactions of matter content.

e.g. (*) $S = -\int d^4x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right]$

1.3.1 Hawking & Unruh temperatures from Euclidean analytic continuation

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H}$$

$$\text{with } Z_\beta = \mathrm{Tr}\ e^{-\beta H} = \mathrm{Tr}\left(e^{-iHt/\hbar}\right)$$

$$t = -i\beta\hbar$$

$$ds^2 = -dt^2 + d\vec{x}^2 \qquad t \to -i\tau \qquad \tau \sim \tau + \beta\hbar \qquad ds_E^2 = d\tau^2 + d\vec{x}^2 \quad (1)$$

Thermal equilibrium at $T = 1/\beta$ described by path integrals in (I) with periodicity $\beta\hbar$

(A) Hawking temp: take BH metric

$$t \to -i\tau$$

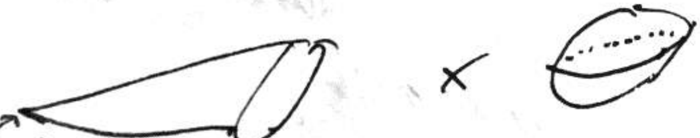
$$\begin{aligned} ds_E^2 &= f\, d\tau^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \\ &= \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_s^2 d\Omega^2 \\ &= \underbrace{\rho^2 d\theta^2 + d\rho^2}_{\text{locally } \mathbb{R}^2} + r_s^2 d\Omega^2 \end{aligned}$$

$\theta = \kappa\tau$ is like an angular variable

Global structure: depends on periodicity of θ

$\theta \sim \theta + 2\pi \Rightarrow$ globally $\mathbb{R}^2$

other periodicity:



$\rho = 0$

$\rightsquigarrow$ ALE? gravitational instantons

$\rho = 0 \Rightarrow$ conical singularity

smoothness of Euclidean geometry

$\Rightarrow \tau \to \tau + \frac{2\pi}{\kappa}$ (uniquely determined)

Different from on Mink.

$$R \times \mathbb{R}^3 \to S^1 \times \mathbb{R}^3$$

any period allowed

$\Rightarrow$ in a black hole geometry, quantum matter can be in equilibrium only at a single temperature $T_H = \frac{1}{\beta_H}$

$$\hbar \beta_H = \frac{2\pi}{\kappa} \Rightarrow T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{8\pi G M}$$

Remarks:

1) T_H should be considered as temperature measured in ~~proper~~ units of proper time at $r = \infty$.

2) $\Rightarrow$ BH must have temperature T_H

3) Field theory on a cone
$\Rightarrow$ observables can be singular at the singularity.

4) suppose $\tau \sim \tau + \beta\hbar \quad \beta \neq \beta_H$

must be singular to screen difference

$T \neq T_H$

$\rho = 0 \Leftrightarrow$ horizon

"force" the equilibrium

5) You can put any matter at any T outside the black hole (including nothing, $T=0$)

$\rightsquigarrow$ non-eq. state

but euclidean A.C. you can only desc. the equilibrium state.

6) $ds^2 = g(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2$

$r = r(T^2 - X^2)$ $\rightsquigarrow$ $T_E^2 + X^2 =$

$T \to -iT_E$

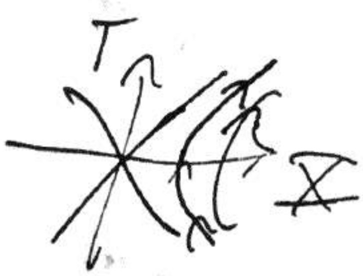
7) For a stationary observer at r

$$t_{loc} = \sqrt{f(r)}\, dt$$

$$\Rightarrow T_{loc} = \frac{\hbar \kappa}{2\pi} f^{-1/2}(r) \quad (2)$$

$r \to r_s$ $\quad T_{loc} \to \infty$

B. Unruh temp



$$ds^2 = -dT^2 + dX^2$$

$\rightsquigarrow$ Rindler space $\quad ds^2 = -\rho^2 d\eta^2 + d\rho^2$

$$\eta \to -i\theta$$

$$ds_E^2 = \rho^2 d\theta^2 + d\rho^2$$

Smoothness of Euclidean space

$\Rightarrow \theta \sim \theta + 2\pi$

local time: $dt_{loc} = \rho \, d\eta$

$d\tau_{loc} = \rho \, d\theta$

$\tau_{loc} \sim \tau_{loc} + 2\pi\rho \; \leftarrow \hbar \beta_{loc}$

$\Rightarrow T_u(\rho) = \frac{\hbar}{2\pi\rho} = \frac{\hbar a}{2\pi}$ (3), $a = 1/\rho$

$\Rightarrow$ a uniformly accelerated obs. in Mink can be in thermal equilibrium only at $T_u(\rho)$, otherwise one finds singular behavior at $T = \pm X$ $(\rho = 0)$

Remarks:

1) (2) and (3) agree when $r \approx r_s$, as expected

2) BH: $r \to \infty$ $T \to T_H$ $(a_{prop} \to 0)$
Rindler: $\rho \to 0$ $T \to 0$ $(a_{prop} \to 0)$

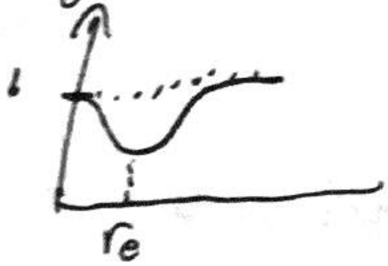
2) Does this happen to all accelerated observers?

$$ds^2 = -g(r)\,dt^2 + \frac{1}{g(r)}\,dr^2 + r^2 d\Omega_2^2$$

$$g(r) = 1 - \frac{2G_N m(r)}{r}$$

$$m(r) = \begin{cases} M_{earth} & r > r_e \\ \propto r^3 & r < r_e \end{cases}$$

$\Rightarrow g(r)$



$t \to -i\tau$ $\quad$ τ can have any periodicity

Hawking, Unruh require $g_H \to 0$

3) Rindler $\quad T \neq T_u$

singular behavior at $T = \pm X$

1.3.2 Unruh temp. from entanglement

1) Clarifies physical origin of temp.

2) Gives deeper understanding of the quantum state of matter

Ai digression — an alternative (Lorentzian) way to describe thermal states

$\mathcal{H}$, H, E_n, $|n\rangle$, $\rho = \frac{1}{Z_\beta} e^{-\beta H}$

$\Rightarrow$ double the system

$$\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H}$$

typical state:

$$\underbrace{\sum_{m,n} a_{mn} |m\rangle_1 \otimes |n\rangle_2}_{\text{non-factorizable}}$$

$$\neq \Psi_1 \otimes \Psi_2$$

$\Rightarrow$ entangled

normalized

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} |n\rangle_1 \otimes |\tilde{n}\rangle_2 \qquad \langle \Psi_\beta | \Psi_\beta \rangle = 1$$

$|\tilde{n}\rangle$ is T-reversal of $|n\rangle$

$Z_\beta = \mathrm{Tr}(e^{-\beta H})$ $\Leftarrow$ either system

Consider X_1 which acts only on $\mathcal{H}_1$

$$\Rightarrow \langle \Psi_\beta | X_1 | \Psi_\beta \rangle = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} \langle n | X | n \rangle$$

$$= \mathrm{Tr}(\rho_B X)$$

$$\mathrm{Tr}_2 (|\Psi_\beta\rangle \langle \Psi_\beta|) = \rho_B$$

$|\Psi_\beta\rangle$: <u>thermal field double</u>

<u>Umezawa</u> (1968?)

<u>Remarks</u>: Finite-T effects come from:

1) Special entangled structure of Ψ_β
2) Ignorance of the other system
3) Purification of ρ_β
4) $(H_1 - H_2)|\Psi_\beta\rangle = 0 \Rightarrow e^{-i(H_1 - H_2)t} |\Psi_\beta\rangle = |\Psi_\beta\rangle$
5) $H = \hbar\omega (a^\dagger a + \frac{1}{2})$ for harmonic oscillator

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

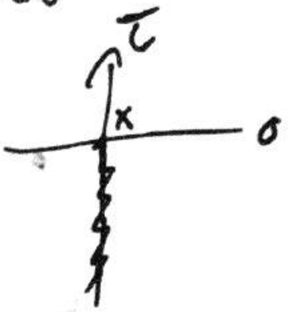
$$\Rightarrow |\Psi_\beta\rangle = \frac{e^{-\frac{1}{4}\beta\hbar\omega}}{\sqrt{Z_\beta}} e^{-\frac{1}{2}\beta\hbar\omega a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

Recall: Path integral for vacuum state

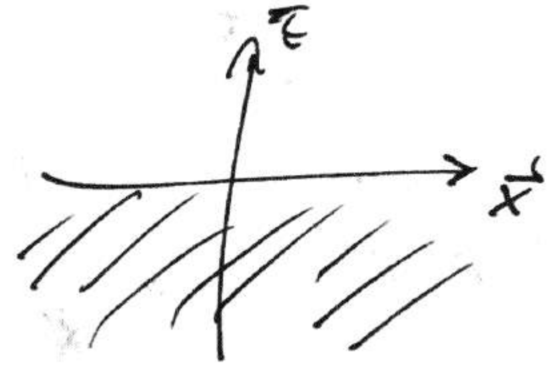
$\psi(x) = \langle x | 0 \rangle$ $\qquad t \to -i\tau$

$$= c \int_{\substack{x(\tau_0)=0 \\ \tau_0 \to -\infty}}^{x(\tau=0)=x} DX(\tau)\, e^{-S_E[x(\tau)]/\hbar}$$

$$= \lim_{\tau \to -\infty} \langle x | e^{\tau H} | 0 \rangle$$

$=$ 

$$ds^2 = -dt^2 + d\vec{x}^2$$
$$\downarrow$$
$$ds_E^2 = d\tau^2 + d\vec{x}^2$$



$$\Psi_0[\varphi(\vec{x})] = \langle \varphi(\vec{x}) | 0 \rangle$$
$$= c \int_{\varphi(\tau \to -\infty) \to 0}^{\varphi(\tau=0,\vec{x})=\varphi(\vec{x})} D\varphi\, e^{-S_E[\varphi]/\hbar}$$

$${}\langle x_2 | \tilde{n} \rangle_2 = {}_2\langle n | x_2 \rangle = {}_1\langle n | x_2 \rangle$$

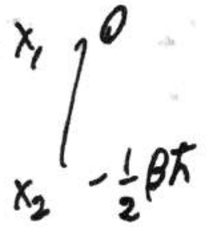
$$\psi_\beta(x_1, x_2) = \langle x_1\, x_2 | \psi_\beta \rangle$$
$$= \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} \langle x_1 | n \rangle_1 \langle x_2 | \tilde{n} \rangle_2$$

'' $= \sum_n e^{-\frac{1}{2}\beta E_n} \langle x_1|n\rangle\langle n|x_2\rangle$

$= \langle x_1| e^{-\frac{1}{2}\beta H} |x_2\rangle$

$= \langle x_1| e^{-\frac{iH}{\hbar}\Delta t} |x_2\rangle \Big|_{\Delta t = -\frac{i}{2}\hbar\beta}$

$$\rightsquigarrow = \frac{1}{\sqrt{Z_\beta}} \int_{x(-\frac{1}{2}\beta\hbar)=x_2}^{x(0)=x_1} Dx(\tau)\, e^{-\frac{1}{\hbar} S_E[x(\tau)]}$$

= 

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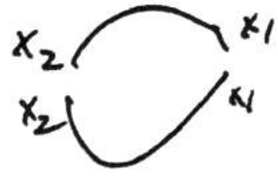
$$\Rightarrow \langle \psi_\beta|\psi_\beta\rangle = \frac{1}{Z_\beta} \times \int_{x(-\frac{1}{2}\beta\hbar)=x_2}^{x(0)=x_1} Dx(\tau) \int_{\tilde{x}(\frac{1}{2}\beta\hbar)=x_2}^{\tilde{x}(0)=x_1} D\tilde{x}(\tau)\, e^{-S_E[x]}$$

assume S_E invariant under
$\tau \to -\tau$

$$= \frac{1}{Z_\beta} \int_{x(-\frac{1}{2}\beta\hbar)=x(\frac{1}{2}\beta\hbar)} Dx(\tau)\, e^{-S_E[x]} = 1$$

= 

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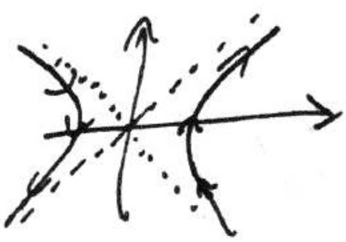
Field theory:

$$\langle \varphi_1(\vec{x})\, \varphi_2(\vec{x}) | \Psi_\beta \rangle = \frac{1}{\sqrt{Z_\beta}} \int_{\varphi(-\frac{1}{2}\beta\hbar,\vec{x}) = \varphi_2(\vec{x})}^{\varphi(0,\vec{x}) = \varphi_1(\vec{x})} D\varphi(\tau,x)\, e^{-S_E[\varphi]} \qquad (*)$$

B. Unruh temperature from entanglement

$$ds^2 = -dT^2 + dX^2 = -\rho^2 d\eta^2 + d\rho^2 \qquad X = \rho\cosh\eta,\quad T = \rho\sinh\eta \quad \leftarrow R \text{ patch}$$



$\eta \to \eta + \text{const} \Rightarrow$ boost in (T, X)

similarly:

$$X = -\rho\cosh\eta,\quad T = -\rho\sinh\eta \quad \leftarrow L \text{ patch}$$

R, L are causally disconnected

Three sets of observers:

Mink: "see" the entire Mink space

Rind_R: ...... R patch

Rind_L: ...... L patch

Mink : Cauchy slice : $T=0$

$\mathcal{H}_{Mink}$: span$\{|\varphi(\vec{x})\rangle\}$

$\varphi(\vec{X}) = \varphi(T=0, \vec{X})$

H_M : using T as time

$|0\rangle_M$

Rind$_R$: Cauchy slice $\eta=0$ ($X>0$ axis)

$\mathcal{H}_{Rind_R}$: span$\{|\varphi_R(\rho)\rangle\}$

$\varphi_R(\rho) = \varphi(T=0, X=\rho>0\}$

H_R : obtained from S restricted to R with η as time

$|0\rangle_R$

Rind$_L$: Cauchy slice : $\eta=0$ ($X<0$ axis)

$\mathcal{H}_{Rind_L}$: span$\{|\varphi_L(\rho)\rangle\}$

$\varphi_L(\rho) = \varphi(T=0, X=-\rho<0)$

Since $\phi(X) = (\phi_L(\rho), \phi_R(\rho))$

$$|\phi(X)\rangle = |\phi_L(\rho)\rangle \otimes |\phi_R(\rho)\rangle$$

$$\Rightarrow \mathcal{H}_{MinK} = \mathcal{H}_{Rind_L} \overset{\overset{PT}{\sim}}{\otimes} \mathcal{H}_{Rind_R}$$

Question:
is $|0\rangle_M$ equivalent to $|0\rangle_L \otimes |0\rangle_R$?

Answer:
It turns out, no.

Note: any field theory is CPT-invariant.

$$\Rightarrow \mathcal{H}_R \overset{\sim}{\leftrightarrow} \mathcal{H}_L$$

(R, L) form a double

Claim: $|0\rangle_M$ is a TFD for $\mathcal{H}_{Rind_L} \otimes \mathcal{H}_{Rind_R}$

strategy for proof: coordinate space ~~~~ wavefunction

Note: (T_E, X)-LHP in fact coincides with Euclidean analytic continuation of Rindler

$$\eta \to -i\theta$$
$$\theta \in (-\pi, 0)$$

$$\Rightarrow \Psi_0[\varphi(x)] = \int_{\varphi(\theta=-\pi,\rho)=\varphi_L(\rho)}^{\varphi(\theta=0,\rho)=\varphi_R(\rho)} D\varphi(\theta,\rho)\, e^{-S_E[\varphi]} \qquad (**)$$

compare $(*)$ w/ $(**)$

$$\Rightarrow \Psi_0[\varphi(x)] = \langle \varphi_R(\rho)\, \varphi_L(\rho) \,|\, \Psi_\beta \rangle$$

with $\frac{1}{2}\beta\hbar = \pi$

$$\Rightarrow \beta = \frac{2\pi}{\hbar}$$

$$\Rightarrow Z_0^{(\text{Mink})} = Z^{(\text{Rind})}_{\beta=\frac{2\pi}{\hbar}}$$

We conclude:

$$|0\rangle_M = |\Psi_{\beta=\frac{2\pi}{\hbar}}\rangle$$

$$Z_0 = \mathrm{Tr}\left(e^{-\frac{2\pi}{\hbar} H_{\text{Rind}}}\right)$$

since β is associated with η

$$dt_{loc} = \rho\, d\eta$$

$$\Rightarrow \boxed{\beta_{loc} = \frac{2\pi\rho}{\hbar}}$$

just as we derived last time but this time from real-time wavefunction

Sept 24 (Missed, got notes from Sam)

Remarks

1) Euclidean method: regularity of analytic continuation

$\Rightarrow$ only have equilibrium at T_u

Now: When system is at $|0\rangle_M$

$\nRightarrow$ R/L observers both thermal at T_u

$$\underset{\substack{\uparrow \\ \text{Mink}}}{Z_0} = \underset{\substack{\uparrow \\ \text{Rind}}}{Z_{\beta = 2\pi/\hbar}}$$

2) Thermal nature comes from

(a) Special entangled structure of $|0\rangle_M$

(b) Tracing out the other half

3) Both derivations used a simple geometric feature:

Euclidean analytic cont. of Mink_2

$\parallel$

Euclidean analytic cont. of Rind + special periodicity

$\rightsquigarrow$ This is very general, applies to any QFT

4) Entanglement method: no need to deal w/ conical singularity

5) $(H_L - H_R)|\Psi_\beta\rangle = 0 \Rightarrow e^{-i\eta(H_L - H_R)}|\Psi_\beta\rangle = |\Psi_\beta\rangle$

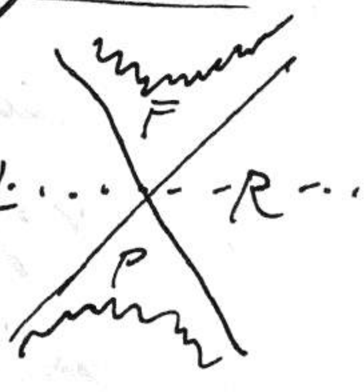
Boost inv. of vacuum $\rightarrow e^{-i\eta(H_L - H_R)}|0\rangle_M = |0\rangle_M$

$H_R = \int_0^\infty dX\, X\, T_{00}$ (on $T=0$ Cauchy slice)

$H_L = \int_{-\infty}^0 dX\,(-X)\,T_{00}$ (on ")

C: Hawking Temperature from entanglement

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$
$$= g(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$$
$$X^2 - T^2 = \frac{1}{\kappa^2} e^{\frac{r - r_s}{r_s}} \left(\frac{r - r_s}{r_s}\right)$$



Similarity: Kruskal observers $\rightarrow T$

Schwarzschild observers $\rightarrow t$

$$\mathcal{H}_K = \mathcal{H}_L \otimes \mathcal{H}_R$$

Important difference from Rindler
$\rightarrow$ Metric is not T-independent $\Rightarrow H_K$ is not either $\Rightarrow$ energy not conserved

no notion of vacuum state

Nevertheless we can define the counterpart of $|0\rangle_M$ using path integrals to get $|0\rangle_{HH}$ "Hartle-Hawking vacuum"

Key:
(1) Kruskal metric allows a sensible $T \to -iT_E$
(2) Euclidean manifold is again the same as that obtained by taking $t \to i\tau$ with $\tau \sim \tau + \frac{2\pi}{\hbar\varkappa}$

$\Rightarrow$ (def) $|0\rangle_{HH} :=$ path integral over $T_E < 0$

$|0\rangle_{HH} = |\Psi_{\beta_H}\rangle \leftarrow$ Thermal field double with $\beta_H = \frac{2\pi}{\hbar\varkappa}$

D: Geometry & Entanglement

Previously: From perspective of Rindler or Schwarzschild observers, there is a singular behaviour at the horizon unless they are at T_H

$\rightarrow$ explain this using entanglement

Rindler: $\eta \to -i\theta$, zero temp $\to$ θ not compact $\to$ $|0\rangle_R \otimes |0\rangle_L$ (*)
in this state L and R are unentangled

For any smooth wavefunction of any QFT in Mink_2

L | R $\rightarrow$ always entangled for any finite energy state

For L, R not entangled, we'd need a barrier at $X=0$
$\Rightarrow$ singular behavior at $X=0$ that causally propagates

Note (*) is not a state in $Mink_2$

$|L\rangle \otimes \langle R| \leftrightarrow$ No FP regions

$\Rightarrow$ Presence of F/P regions in $Mink_2$

$\Updownarrow$

Entanglement of L, R.

Generalize:

1) Any finite energy state in $Mink_2$ with all 4 regions requires specific entanglement between L and R

2) Generic state in $\mathcal{H}_L \otimes \mathcal{H}_R$ doesn't have that entanglement structure

$\Rightarrow$ Cannot be interpretted as sensible in $Mink_2$

3) Similarly, generic state in $\mathcal{H}_L \otimes \mathcal{H}_R$ for schwarzschild will have "fire wall" at the horizon

4) discussion is at the level of states, independent of details about $\mathcal{H}_M$, $\mathcal{H}_L$, $\mathcal{H}_R$, etc.

1.3.3 Free Field Theories Derivation

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} |n\rangle_L \otimes |n\rangle_R = \frac{e^{-\frac{1}{4}\beta\hbar\omega}}{\sqrt{Z_\beta}} e^{e^{-\frac{1}{2}\beta\hbar\omega} a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

Note $\quad a\, e^{\alpha a^\dagger}|0\rangle = \alpha e^{\alpha a^\dagger}|0\rangle \qquad a_1|0\rangle_1 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a_2|0\rangle_2 = 0$

$$\Rightarrow \left(a_1 - e^{-\frac{1}{2}\beta\hbar\omega} a_2^\dagger\right)|\Psi_\beta\rangle = 0$$

$$\left(a_2 - e^{-\frac{1}{2}\beta\hbar\omega} a_1^\dagger\right)|\Psi_\beta\rangle = 0$$

def: $\quad b_1 = \cosh\theta\, a_1 - \sinh\theta\, a_2^\dagger$

$\qquad\quad b_2 = \cosh\theta\, a_2 - \sinh\theta\, a_1^\dagger$

$\Rightarrow$ take $\cosh\theta = \dfrac{1}{\sqrt{1 - e^{-\beta\hbar\omega}}}$, $\sinh\theta = \dfrac{e^{-\frac{1}{2}\beta\hbar\omega}}{\sqrt{1 - e^{-\beta\hbar\omega}}}$

then $\quad [b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$

$\qquad$ else $[\cdot,\cdot] = 0$

and $\quad b_1|\Psi_\beta\rangle = b_2|\Psi_\beta\rangle = 0$

$\Rightarrow |\Psi_\beta\rangle$ is vacuum for b_1, b_2

Free Field theories $\rightarrow$ a bunch of harmonic oscillators

"$|0\rangle_1, |0\rangle_2$" $\rightarrow$ "$|0\rangle_L\ |0\rangle_R$", "$|\Psi_\beta\rangle$" $\rightarrow$ "$|0\rangle_m$"

Free Massless scalar:

$$S = -\frac{1}{2}\int d^2x\, \partial^\mu \varphi\, \partial_\mu \varphi$$

Minkowski observers

$$(-\partial_T^2 + \partial_X^2)\varphi = 0$$

$$\Rightarrow u_p = \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p T + ipX}$$

$$\omega_p = |p|$$

$$U = T - X,\quad V = T + X \Rightarrow u_p = \begin{cases} \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p U}, & p>0 \\ \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p V}, & p<0 \end{cases}$$

$$(\varphi_2, \varphi_1) := i\int_{-\infty}^{\infty} \left(\varphi_2^* \partial_T \varphi_1 - (\partial_T \varphi_2^*)\varphi_1\right)$$

$$\underbrace{\begin{aligned}(u_p, u_{p'}) &= 2\pi\,\delta(p-p') \\ (u_p^*, u_{p'}^*) &= -2\pi\,\delta(p-p') \\ (u_p, u_{p'}^*) &= 0\end{aligned}}$$

CCR with $\varphi = \sum_p (a_p u_p + a_p^\dagger u_p^*)$

$$[a_p, a_p^\dagger] = 2\cancel{\pi}\,\delta(p-p')$$

$$a_p |0\rangle_M = 0$$

Rindler "R"-observers

$$ds^2 = \underbrace{-dT^2 + dX^2}_{\text{Mink}} = -\rho^2 d\eta^2 + d\rho^2$$

$$\rho = e^{\xi} \Rightarrow ds^2 = e^{2\xi}\overbrace{(-d\eta^2 + d\xi^2)}^{\text{Rind}}$$

$$(-\partial_\eta^2 + \partial_\xi^2)\varphi = 0$$

$$u = \eta - \xi,\quad v = \eta + \xi \Rightarrow v_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k v} \end{cases}$$

$$U = -e^{-u},\quad V = e^{v}$$

$$\varphi_R = \sum_k \left(b_k^{(R)} v_k + b_k^{(R)\dagger} v_k^*\right)$$

$$[b_k, b_k^\dagger] = \underset{\text{metric?}}{2\pi}\,\delta(k-k')$$

$$b_k^{(R)} |0\rangle_R = 0$$

Sept 26 (Missed, got notes from Sam)

Recall $U = -e^{-u}$, $V = e^{v}$ (So $U<0$, $V>0$)

$\Rightarrow$ Rindler "R" modes become

$$V_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k v} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k \log(-U)} & k>0 \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log(V)} & k<0 \end{cases}$$

$$\varphi_R = \sum_k b_k^{(R)} V_k + b_k^{(R)\dagger} V_k^* \Rightarrow [b_k^{(R)}, b_{k'}^{(R)\dagger}] = 2\pi\delta(k-k')$$

Define right vacuum $\boxed{b_k^{(R)} |0\rangle_R = 0}$ Note $(\varphi_R^{(2)}, \varphi_R^{(1)}) = i\int_{-\infty}^{\infty} d\xi (\varphi_R^{(2)*} \partial_\eta \varphi_R^{(1)} - (\partial_\eta \varphi_R^{(2)})^* \varphi_R^{(1)})$

V_k is singular at $U=0$ and $V=0$ since the modes are only supported in the R region

$\Rightarrow$ potential singular behavior in physical quantities

Rindler "L":

$T = -e^{\xi} \sinh\eta$, $X = -e^{\xi}\cosh\eta$, $U = e^{-u}_{>0}$, $V = -e^{v}_{<0}$

$$v_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k v} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega \log(U)} & k>0 \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega \log(-V)} & k<0 \end{cases}$$

we have chosen positive-frequency modes to be the PT reversed of V_k

PT reversal ensures $k>0$ has u, $k<0$ has v

Note: $(\varphi_L^{(2)}, \varphi_L^{(1)}) = -i\int_{-\infty}^{\infty} d\xi (\varphi_L^{(2)*} \partial_\eta \varphi_L^{(1)} - (\partial_\eta \varphi_L^{(2)})^* \varphi_L^{(1)})$.

(recall $|\Psi_\beta\rangle = \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle_L |n\rangle_R$

So $\varphi_L = \sum_k \left(b_k^{(L)} w_k + b_k^{(L)\dagger} w_k^*\right) \rightarrow [b_k^{(L)}, b_{k'}^{(L)\dagger}] = 2\pi\delta(k-k')$

Left vacuum $\boxed{b_k^{(L)} |0\rangle_L = 0}$

On $Mink_2$, $\Phi(T, X) = (\varphi_L, \varphi_R)$

$$\Rightarrow \varphi = \sum_p \left(a_p u_p + a_p^\dagger u_p^*\right) = \sum_k \left(b_k^L w_k + b_k^{L\dagger} w_k^* + b_k^R v_k + b_k^{R\dagger} v_k^*\right)$$

To find relation between $\{a_p, a_p^\dagger\}$ and $\{b_k^L, b_k^{L\dagger}, b_k^R, b_k^{R\dagger}\}$, need to find relations between $\{u_p, u_p^*\}$ and $\{w_k, w_k^*, v_k, v_k^*\}$

Two possibilities:

(1) $v_k = \sum_p c_{kp} u_p$, $w_k = \sum_p \tilde{c}_{kp} u_p$ without u^* involved

$\Rightarrow$ positive frequency modes of both L and R observers are related to only positive freq. Minkowski modes

$\Rightarrow b_k^{(R)} = \sum_p d_{pk} a_p$, $b_k^{(L)} = \sum_p \tilde{d}_{pk} a_p$

$\Rightarrow |0\rangle_M$ coincides with $|0\rangle_L \otimes |0\rangle_R$

(2) suppose $u_p = \sum_k \left(d_{pk} v_k + \tilde{d}_{pk} w_k + e_{pk} v_k^* + \tilde{e}_{pk} w_k^*\right)$

then $b_k^{(R)} = \sum_p \underbrace{\left(d_{pk} a_p + e_{pk}^* a_p^\dagger\right)}$, $b_k^{(L)} = \sum_p \underbrace{\left(\tilde{d}_{pk} a_p + \tilde{e}_{pk}^* a_p^\dagger\right)}$

Bogoliubov Transformations

$\Rightarrow \boxed{|0\rangle_M \neq |0\rangle_L \otimes |0\rangle_R}$

Requiring $b_k^{(R)} |0\rangle_L \otimes |0\rangle_R = b_k^{(L)} |0\rangle_L \otimes |0\rangle_R = 0$

we get: $|0\rangle_L \otimes |0\rangle_R \sim e^{\# a^\dagger a^\dagger} |0\rangle_M$

$$|0\rangle_M \sim e^{\# b^\dagger b^\dagger} |0\rangle_L \otimes |0\rangle_R$$

Focus on right-moving modes:

$u_p = \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p U}$, $v_p = \begin{cases} \frac{1}{\sqrt{2\omega_p}} e^{i\omega_p \log(-U)} & \text{for } U<0 \\ 0 & \text{for } U>0 \end{cases}$, $w_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log U} & \text{for } U>0 \\ 0 & \text{for } U<0 \end{cases}$

argument for possibility (2):

u_p is analytic in lower complex U plane

$\rightarrow$ So is any linear superposition

Neither v_k or w_k is analytic there

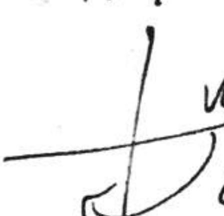
$\Rightarrow$ v_k, w_k must involve both u_p, u_p^*

Instead of finding $d_{pk}, \tilde{d}_{pk}, e_{pk}, \tilde{e}_{pk}$ explicitly (which is doable)

consider another basis equivalent to (u_p, u_p^*) (i.e. has the same vacuum) but related to v, v^*, w, w^* in a simple way:

LU

construct λ_k from analytic cont. of v_k to $U>0$ in lower half plane



analytic continuation in LH plane $\Rightarrow$

$$\lambda_k = \frac{1}{\sqrt{2\sinh \pi\omega_k}} \left[e^{\frac{\pi}{2}\omega_k} v_k + e^{-\frac{\pi}{2}\omega_k} w_k^* \right]$$

$$\chi_k = \frac{1}{\sqrt{2\sinh \pi\omega_k}} \left[e^{\frac{\pi}{2}\omega_k} w_k + e^{-\frac{\pi}{2}\omega_k} v_k^* \right]$$

$\{\lambda_k, \chi_k\}$ share the same vacuum, $|0\rangle_M$ as $\{u_k\}$

$$\Rightarrow \varphi = \sum_k \left[c_k \lambda_k + d_k \chi_k + h.c. \right], \quad c_k |0\rangle_M = d_k |0\rangle_M = 0$$

$$c_k = \cosh\theta_k \, b_k^{(R)} - \sinh\theta_k \, b_k^{(L)\dagger}$$

$$d_k = \cosh\theta_k \, b_k^{(L)} - \sinh\theta_k \, b_k^{(R)\dagger}$$

where
$$\cosh\theta_k = \frac{e^{\frac{1}{2}\pi\omega_k}}{\sqrt{2\sinh\pi\omega_k}}$$
$$\sinh\theta_k = \frac{e^{-\frac{1}{2}\pi\omega_k}}{\sqrt{2\sinh\pi\omega_k}}$$

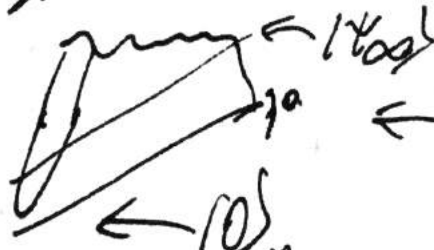
$$\Rightarrow |0\rangle_M = \prod_k \left(\frac{e^{-\frac{\pi}{2}\omega_k}}{\sqrt{Z_k}} \right) \exp\left[\sum_k e^{-\pi\omega_k} \, b_k^{(R)\dagger} b_k^{(L)\dagger} \right] |0\rangle_L \otimes |0\rangle_R$$

with $Z_k = \dfrac{1}{2\sinh\pi\omega_k}$ $\Big\}$ for each k this is exactly the result for a QHO at $\beta = \frac{2\pi}{\hbar}$

(Free) <u>Massive scalar in Schwarzschild background</u>:

$|0\rangle_{HH} \leftarrow$ squeezed state for Schwarzschild obs.

- Realistic BH



$\leftarrow$ get temp. from free field theory

1.4 BH Thermodynamics

BH has a temperature

$$T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{8\pi G_N M} \quad (1) \Rightarrow$$ natural to interpret it as a thermodynamic system

suppose it has an entropy S.

Expect it to satisfy 1st law:

$$dE = T\,dS \quad (2)$$

identify $E = M$, integrate (2) to find S

$$dS = \frac{dM}{T} \Rightarrow \boxed{S = \frac{4\pi G_N M^2}{\hbar}}$$

but $r_s = 2G_N M$

$$\Rightarrow \boxed{S = \frac{4\pi r_s^2}{G_N \hbar} = \frac{A}{4\hbar G_N}} \quad (3)$$

So using (1) and (3), we can rewrite (2) as

$$dM = \frac{\hbar \kappa}{2\pi} \frac{A}{4\hbar G_N} = \frac{\kappa}{8\pi G_N} dA \quad (4)$$

(4) is a pure geometric relation

Eq. (4) is part of a set of four laws on general BHs called "Four laws of BH mechanics"

• No hair theorem:

A stationary asymptotically flat BH is solely characterized by:

1) mass M
2) angular momentum J
3) electric or magnetic charges Q

• Four laws: (1972)

0th law: Surface gravity κ is constant over the horizon

1st law: $dM = \frac{\kappa}{8\pi G_N} dA + \Omega dJ + \Phi dQ$

Ω: angular frequency at the horizon

Φ: electric potential (s.t. $\Phi(\infty)=0$)

2nd law: Horizon area never decreases classically.

3rd law: Surface gravity of a BH cannot be reduced to 0 in a finite sequence of operations

With (1) + (3), the four laws of BH mechanics become the standard laws of thermodynamics

Beckenstein (1972–1974):

BH should have an entropy $\propto A$

otherwise the second law of thermodynamics would be violated in the presence of a BH

Define

$$S_{tot} := S_{matter} + S_{BH}$$

$\rightsquigarrow$ Generalized second law

$$\underline{\Delta S_{tot} \geq 0}$$

with (1975) Hawking radiation, GSL becomes standard 2nd law.

Remarks:

1) classical limit $\hbar \to 0$ (G_N fixed)

$$T_H \to 0, \quad S_{BH} \to \infty$$

2) $T_H \propto \frac{1}{M}$, $M\uparrow \Rightarrow T_H \downarrow$

$$\Rightarrow \underset{\text{specific heat}}{C} = \frac{\partial M}{\partial T} = -\frac{\hbar}{8\pi G_N}\frac{1}{T^2} < 0 \Rightarrow \text{negative specific heat}$$

$\Rightarrow$  $\leftarrow T_H$ not a stable equilibrium. Why? Consider small fluctuations in T_{BH}, say $T_{BH}\uparrow$, radiate a bit more to env $\Rightarrow M\downarrow \Rightarrow T_{BH}\uparrow \Rightarrow$ radiate <u>even more</u> and similarly unstable in the other direction. $\boxed{MB}$

BH + infinite bath

Stable equilibrium is possible in a finite box

3) $T_H = \frac{\hbar \varkappa}{2\pi}$ and $S = \frac{A}{4\hbar G_N}$

- Universal

Apply to any matter coupled to Einstein gravity! (AdS, dS, Mink, all spacetimes)

4) With higher derivative corrections to Einstein gravity

(i.e. $R^2 + \lambda^2 R_{\mu\nu}^2 + \mathcal{R}^2 + \dots$)

these ~~[illegible]~~ equations no longer apply but S, T_H can still be expressed in terms of horizon quantities

1.5 Quantum Nature of Black Holes and the Holographic Principle

BH thermodynamics + $T_H \propto \hbar$
$S_{BH} \propto 1/\hbar$

$\Rightarrow$ Natural to treat BH as a <u>macroscopic</u> <u>quantum statistical system.</u>

Questions:

(1) What is the statistical interpretation of the entropy of a black hole?

From standard Stat Mech:

$\#$ microstates $= \Omega$ (consistent w/ a given macroscopic equilibrium)

$$\Rightarrow S = k_B \log \Omega$$

For BH, expect $\Omega = e^{\frac{A}{4\hbar G_N}} = e^{\frac{A}{4 l_p^2}}$

Heuristically:

entropy ~ put 1 d.o.f. in each Planckian cell.
(e.g. spin)

(a) This is a huge entropy

For $M_{BH} = M_\odot$, $r_S = 3\,\mathrm{km} \Rightarrow \frac{A}{4\hbar G_N} \cong 1.1 \times 10^{77}$

$$\Rightarrow \Omega \approx e^{10^{77}}$$

The sun itself has entropy $\frac{S}{\hbar} \approx 10^{57}$

(b) When a star collapses to form a BH, there is a <u>huge increase</u> in # of available microstates

<u>no hair theorem</u>: all these states must be quantum mechanical in nature.

<u>huge increase</u> $\Leftrightarrow$ gravity is weak (Planck scale is very small)

(c) In string theory, there are black holes whose microstates can be precisely counted giving (after complicated combinatorics), exactly an entropy of $S = \frac{A}{4\hbar G_N}$

(d) In holographic duality, for black holes in AdS, the statistical origin is again known

(2) Hawking's information loss paradox

Hawking:

1) To an excellent approximation BH radiates thermally for $M \gg M_{pl}$
"white noise"

2) BH loses mass

3) should disappear

But when $M \sim M_{pl}$, not enough d.o.f. to encode all the information put into it

Another way to say this:

suppose a star is in a pure state
$\Rightarrow$ BH
$\Rightarrow$ Radiates
$\Rightarrow$ Radiation (mixed (thermal) density matrix)

$\Rightarrow$ 3 logical possibilities:

1) Information is lost $\Rightarrow$ QM must be modified

2) Hawking radiation stops at $M \sim O(M_{pl})$ R $\otimes$ Radiation
$\Rightarrow$ Planckian mass remnant is left, which encodes all information

3) No remnants, unitary evolution $\Rightarrow$ information comes out from radiation

1) Is the most radical. It is also fiendishly difficult to modify QM

2) Blames unknowns ⟸ another universe

A variant:

3) Is the most conservative

⇝ significant challenges still to explain <u>how</u> the information comes out from radiation

⇒ imply quantum gravity puts highly nontrivial constraints/implications on low-energy physics

simpler question: Burning of coal

- Preparation: A <u>typical</u> <u>highly excited</u> <u>pure state</u> in a <u>non-integrable</u> <u>many-body</u> system <u>looks</u> thermal if one only probes a small part of it.

Say I separate the system in two:

$$A + B$$

"trace distance is exponentially in [illegible] to therm"

$$\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$ is very close to being thermal

provided $|A| \lesssim |B|$

[illegible]
3 cont. $\Rightarrow$ One reveals a given state is pure only by having full global information

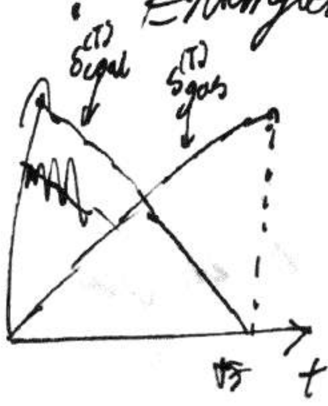
<u>some remarks:</u>

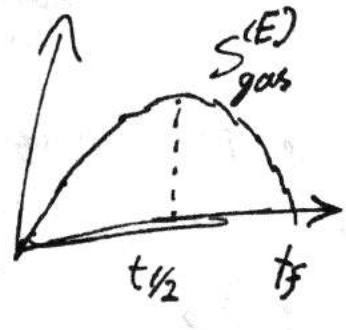
(a) At a given time, emitted photons look almost perfectly thermal.

(b) Nevertheless, they <u>do</u> contain information, but in a very subtle way.
The information is encoded in the entanglement with the rest of the system. $\rightarrow e^{-N} \sim e^{-1/G_N}$ ← nonperturbative

(c) Consider the following quantities:

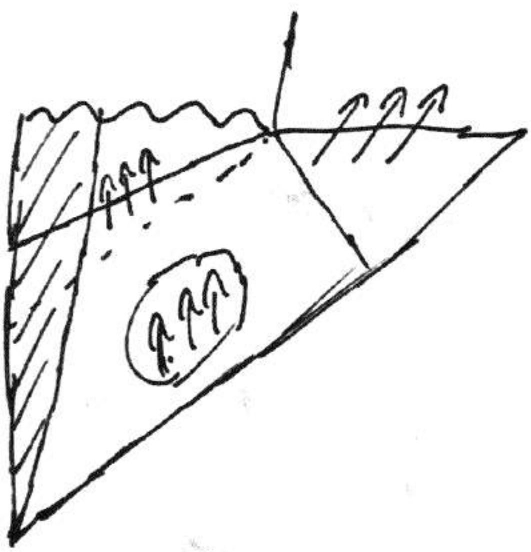
- Thermal entropy of photon gas: $S^{(T)}_{gas}$
- Thermal entropy of the coal: $S^{(T)}_{coal}$ } "coarse-grained"
- Entanglement entropy of photon gas: $S^{(E)}_{gas}$
- Entanglement entropy of the coal: $S^{(E)}_{coal}$ } "fine-grained"





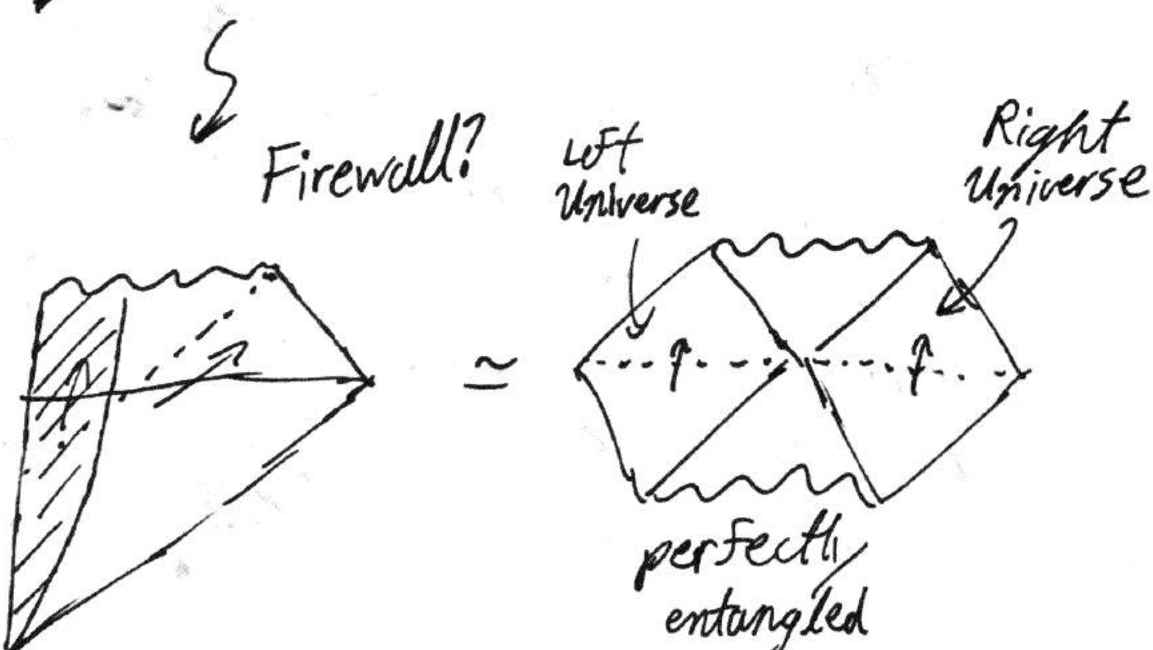
$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

This is a good paradigm for BH evaporation but BH is not coal.
Coal is causally connected with emitted radiation.
Black hole's infalling matter is causally disconnected with the emitted radiation.



Would either violate

- No-cloning theorem (QM)
- Locality (QFT)



Holographic duality tells us that information evaporation for a Black Hole should be just like the burning of coal

· Entropy bounds and holographic principle

starting point:

BH is a quantum statistical system

+ a couple of "facts"

$\Rightarrow$ entropy bounds and holography

Facts:

(1) A sufficiently massive object in a compact volume always collapses to form a Black Hole

Rule of thumb: if $2G_N M > L \Rightarrow$ BH

(2) Entropy reflects # of degrees of freedom

$\rho \rightarrow S = -\mathrm{Tr}[\rho \log \rho]$

$\Rightarrow$ for a system with N-dimensional Hilbert space

$S_{max} = \log N$

(a) for a system of spins $N = 2^n \Rightarrow S_{max} \sim n$

(b) $\mathcal{H}$ for h.o. is infinite-dimensional, but for finite energy, $\mathcal{H}$ is f.d

* Spherical entropy bound

$\Rightarrow$ Take an isolated system of energy E, entropy S_0 in asymptotically flat spacetime

Let A be the area of smallest sphere enclosing the system, M_A be the mass of a black hole with that area.

Then $E \leq M_A$

$\Rightarrow$ Maximal ~~entropy~~ energy one could add (keeping A fixed) is

$M_A - E$

$S_{final} = S_{BH} \geq S_{init}$

$= S_0 + S'$

$$S_0 \leq S_{BH} = \frac{A}{4\hbar G_N} \Rightarrow S_{max} = \frac{A}{4 l_p^2}$$

Remarks:

1) S is ~~classically~~ extensive, $\propto V$

$\Rightarrow$ QG behaves very differently from non-gravitational systems

2) A cubic lattice of spins (size L, spacing a) has $S_{max} = \frac{L^3}{a^3} \log 2$ (*)

At $G_N = 0$

Now slowly increase G_N (i.e. l_p)

Then (*) violates the bound when

$$\frac{L^3}{a^3} \log 2 \gtrsim \frac{A}{4 l_p^2} \quad A \sim L^2$$

$$\frac{l_p^2}{L^2} \gtrsim \frac{a}{L} \cdot * = \lambda$$

Suppose each site has mass m

$$M = \frac{L^3}{a^3} m$$

total system is a Black hole when

$$\hbar \frac{\overset{l_p^2}{\overbrace{G_N}} M}{\hbar} > L \Rightarrow l_p^2 \frac{L^3}{a^3} \frac{m}{\hbar q/l_c} > L$$

$$\Rightarrow \frac{l_p^2}{a^2} > \frac{a}{L}\frac{l_c}{L} = \lambda_2$$

$$\lambda_2 \ll \lambda_1$$

So (a) $S \propto A$

and (b) $S \propto V$

But <u>both</u> conditions are important:

a) In a closed universe S^3

$A = 0$, $S \neq 0$

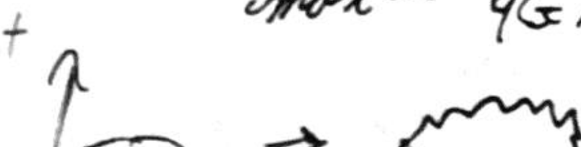
trivially violated

b) consider non-spherical region in asymptotically flat universe

take V space-like, with boundary ∂V

$$S_{max} \overset{?}{=} \frac{A(\partial V)}{4G\hbar}$$

(a)+(b): For a general region, V, with bdy ∂V, in general ∂V has not much to do with physics inside V



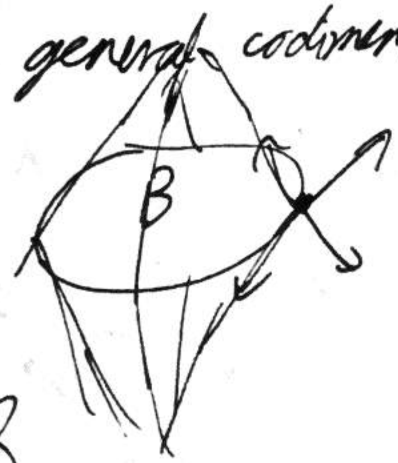
light-like $\Rightarrow A(\partial V) \approx 0$

I don't understand this

A generalization for asymptotic flat spacetimes (universes)

- Consider a general codimension 1 <u>spacelike</u> closed surface

$\Rightarrow$ 4 light rays



Future in
out

past in
out

c.f. hep-th/0203101

causal diamond of B

Take D to denote the causal diamond
- any point in D is fully determined by the information enclosed in B

- conjecture:

$$\text{Entropy on any Cauchy slice of } D \leq \frac{A_B}{4\hbar G_N}$$

This doesn't work in cosmological settings or inside a black hole

Most general formulation:
For any B, construct light-sheet from B: null hypersurface formed by non-expanding light rays from B.

$$\Rightarrow S[L(B)] \leq \frac{A_B}{4\hbar G_N}$$

entropy of "matter" d.o.f. passing through light sheet.

Entropy bound $\leftarrow$ entropy associated with # d.o.f.

$\Rightarrow$ statement on # of d.o.f.

$\Rightarrow$ Holographic principle

"When something is so new, that you don't have the language to describe it, you describe it in whatever language you can. It may not be precise, it may not even be true, but it's better than nothing."

~ Xiao-gang Wen

$\rightsquigarrow$ A spherical region of boundary area A can be fully described by no more than

$$\frac{A}{4\hbar G_N} = \frac{A}{4 l_p^2} \text{ d.o.f.}$$

i.e. ~ one degree of freedom per planck-area

Chapter 2: Matrices & Strings

2.1 Path integrals of strings

. QFT: a theory of "particles"
If particle interactions are weak, we can consider the first quantized approach

$$X^\mu(\tau) \qquad \mu = 0, 1, \ldots, d-1$$

$$S_{particle} = -m \int \underbrace{dl}_{\text{proper length}}$$

$$= -m \int d\tau \frac{dl}{d\tau} = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \cdot \frac{dX^\nu}{d\tau}} \quad (*)$$

1) Lorentz invariant
2) Correct equation of motion
3) Correct non-relativistic limit
4) Reparameterization invariant in τ

$$\tau \to \tau'(\tau)$$
$$X^\mu \to X^{\mu'} \quad \text{s.t. } X^\mu(\tau) = X^{\mu'}(\tau')$$

For a quantum particle:

$$G(X,X') = \sum_{\substack{\text{path} \\ \text{from } X \text{ to } X'}} e^{iS_{\text{particle}}}$$ but "$\sqrt{\ }$" is awkward to deal with

rewrite S_{particle} as

$$S = \frac{1}{2}\int d\tau \left(e^{-1}(\tau)\,\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - e(\tau)\,m^2\right)$$

Upon eliminating e, we go back to $(*)$

$$\Rightarrow G(X,X') = \int_{X_i^\mu = X'}^{X_f^\mu = X} DX^\mu\, De\; e^{iS_{\text{particle}}}$$

$\Rightarrow$ = Feynman propogator for a scalar field of mass m

Note:

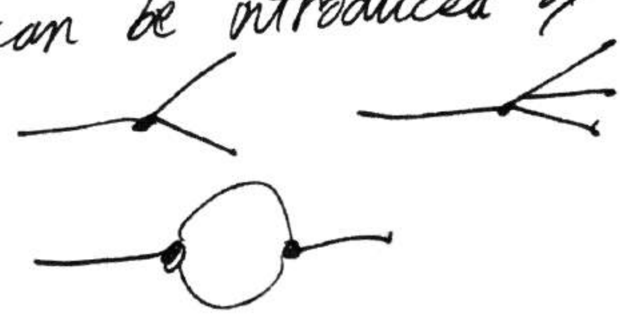
1) $e(\tau)$ is an intrinsic "vielbein" along the worldline

$\underline{\underline{h_{\tau\tau}}} = -e^2(\tau)$ ← intrinsic metric on the worldline

2) For curved spacetime, simply do

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(X(\tau))$$

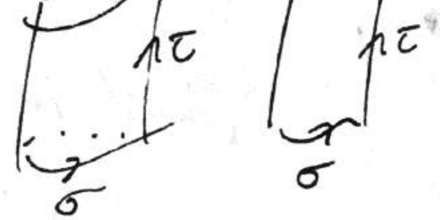
3) Interactions can be introduced by including vertices



4) No general principle to restrict allowed types of vertices $\Leftrightarrow$ interactions in a QFT have to be specified by hand

• Strings:

1-d objects

world sheet 

$$\Sigma: \; X^\mu(\xi^a) \qquad \begin{aligned} a &= 0,1 \\ \xi^a &= (\sigma, \tau) \end{aligned}$$

(*) can be immediately generalized

$$S_{NG} = -T \int_\Sigma dA$$

$$= -T \int_\Sigma d^2\xi \, \sqrt{-\det\Big(\underbrace{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b}}_{h_{ab}\ \text{\underline{induced metric}}}\Big)}$$

$$= -T \int_\Sigma d^2\xi \, \sqrt{-h}$$

Alternative form:

Introduce new auxiliary metric γ^{ab} "intrinsic metric on Σ"

$$S_p = -\frac{T}{2}\int_\Sigma d^2\xi\, \sqrt{-\gamma}\,\gamma^{ab} h_{ab}$$

Quantum dynamics of string:

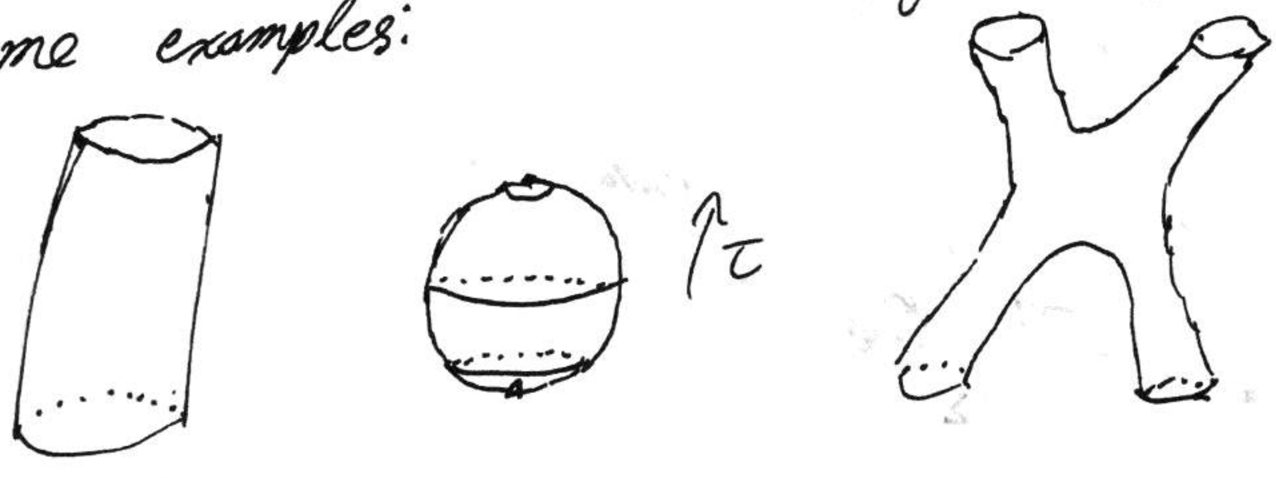
Integrate over all possible string trajectories

$\Leftrightarrow$ Integrate over all 2-d surfaces (weighted by e^{iS_p})

(hard to draw)

$$A = \int DX(\xi^a)\, D\gamma_{ab}\, e^{iS_p}$$

Some examples:

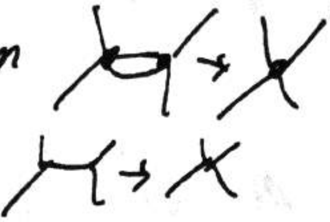


Remarks:

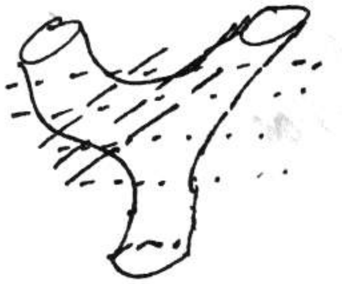
(a) Each such string diagram should be understood as representing integration of all continuous deformations of the corresp. surface, i.e. view each diagram as a rubber sheet (integrating over γ_{ab}, X^μ corresponds to arbitrary stretching)

b) Stretching a 2-d surface is much richer than stretching a line, leading to many important new features of string theory

(1) No sharp interacting vertices

, UV divergences come from 

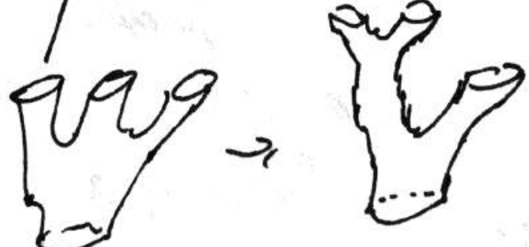
vs

 $\Rightarrow$ string theory has excellent UV behavior (in fact UV finite)

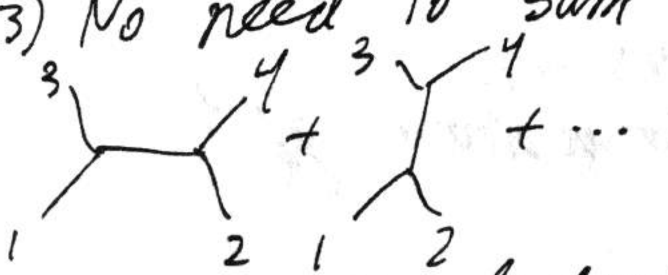
(2) Interactions of strings are essentially unique:
All 2-D surfaces can be built up from:

$\underline{g_s}$ ← key fundamental constant in string theory

For particles,  cannot be reduced to 

but for strings 

(3) No need to sum over different channels



vs 

$\Rightarrow$ Tree level scattering of strings has a single diagram

c) Diagrams of different loops are given by ~~[illegible]~~ surfaces of different topologies, which cannot be continuously deformed to one another

d) Basic theorem of topology:

closed Orientable 2-d surfaces are classified by a single integer h called genus, which is the number of holes (handles)

genus 0: 

1: 

2: 

genus = # of loops

sum over loops = sum over topologies

$\Rightarrow$ at each loop order we have a single diagram

- g_s-dependence $\Rightarrow$ at each loop multiply by g_s^2 $\Rightarrow$ for h-loops g_s^{2h}

tree-level amplitude for n strings: g_s^{n-2}

$$A_n = g_s^{n-2}\left(A_n^{(0)} + g_s^2 A_n^{(1)} + \cdots + g^{2h} A_n^{(h)} + \cdots\right)$$
$$= \sum_{h=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

Path integrals over surfaces of genus h with n external strings

also applies to $n=0,1,2$ from unitarity

$n=0$  $\sim g_s^{-2}$

$n=1$  $\sim g_s^{-1}$ $\simeq$  (i)

$A_n^{(0)} =$  $\Rightarrow$ Each external leg can be ~~considered~~ associated with g_s

Note: $2-n-2h = \chi_{n,h}$: Euler character for a surface of h handles and n boundaries.

$$\Rightarrow A_n = \sum_{h=0}^{\infty} g_s^{-\chi_{n,h}} A_{n,h}$$

Open strings are the same story:

 g_o  g_o

  $\Rightarrow$ adding a loop $\Leftrightarrow$ adding a boundary

$\propto g_o^2$

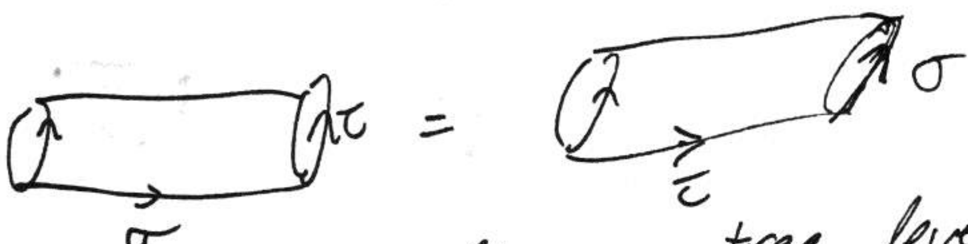
$$A_n^{(open)} = \sum_{L=0}^{\infty} g_o^{\,n-2+2L} A_n^{(L)}$$

$n=0$:
$L=0$

g_o^{-2} (ii)

Now a profound statement:

A theory of open strings must contain closed strings



$\Rightarrow$ 1-loop open string = tree level propagation of closed string

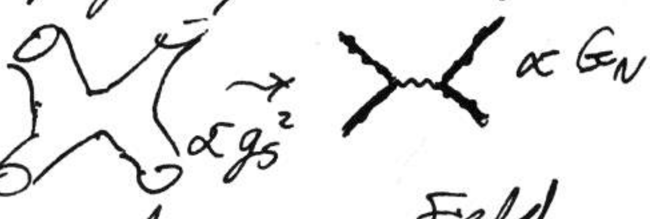
For a theory with both open and closed strings, comparing diagrams (i) and (ii) gives $g_s \propto g_o^2$.

Altogether: A_n contributions go as $g_s^{\,n_c + \frac{n_o}{2} - 2 + 2h + b}$

n_c: # closed strings, n_o: # open strings, h: # handles, b: # boundaries

Closed string excitations: graviton, ...

$\Rightarrow$ gravity $\Rightarrow$ $G_N \sim g_s^2$



Open string excitations: A_μ gauge field

$g_{YM}^2 \propto g_o^2$

2.2 Matrix integrals in the large-N limit

A non-abelian gauge theory with $SU(N)$ gauge group

$$\mathcal{L} = -\frac{1}{4}\frac{1}{g_{YM}^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} - i[A_\mu, A_\nu]$$

$$A_\mu = (A_\mu)^a{}_b \qquad a,b = 1,\dots,N$$

$N\times N$ traceless hermitian matrices

$N=3$: gluon sector of QCD

't Hooft 1974: $N\to\infty$
expand in $1/N$

consider first a matrix scalar theory:

$$\mathcal{L} = -\frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m^2}{2}\Phi^2 + \frac{1}{4}\Phi^4\right]$$

$\Phi = \Phi^a{}_b$ $N\times N$ hermitian

$$(\Phi^a{}_b)^* = \Phi^b{}_a$$

i.e.
$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}\partial_\mu\Phi^a{}_b\,\partial^\mu\Phi^b{}_a + \frac{1}{2}m^2\Phi^a{}_b\Phi^b{}_a + \frac{1}{4}\Phi^a{}_b\Phi^b{}_c\Phi^c{}_d\Phi^d{}_a\right]$$

for any spacetime dimension d

$d=0$: Matrix integral
$d=1$: QM (Matrix)

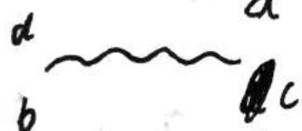
$\mathcal{L}$ is invariant under a $U(N)$ global symmetry

$$\Phi(x) \to U\,\Phi(x)\,U^\dagger$$

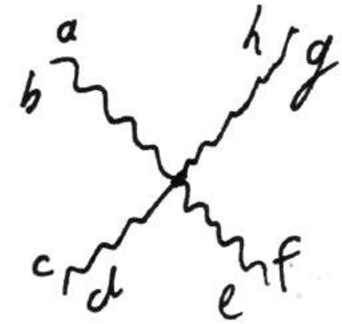
$$U \in U(N) \quad \text{const.}$$

Feynman rules:

$$\langle \Phi^a{}_b(x)\,\Phi^c{}_d(y)\rangle$$

$=$ 

$= g^2\, \delta^a_d\, \delta^c_b\, G(x-y)$ ← scalar propagator



$$= \frac{1}{g^2}\, \delta^a_h\, \delta^c_b\, \delta^e_d\, \delta^g_f$$

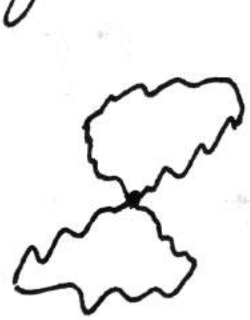
vacuum process:

$$Z = \int D\Phi(x)\, e^{i\int d^dx\, \mathcal{L}}$$

$\log Z$ = sum of all connected diagrams with no external legs

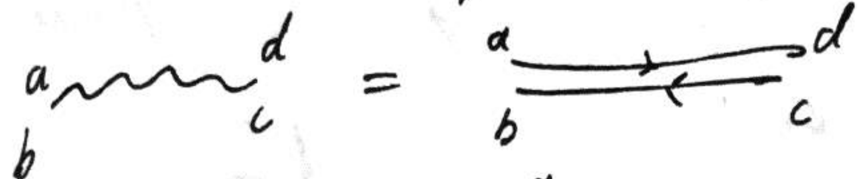
(a) $\propto g^2 N^3$ (planar)

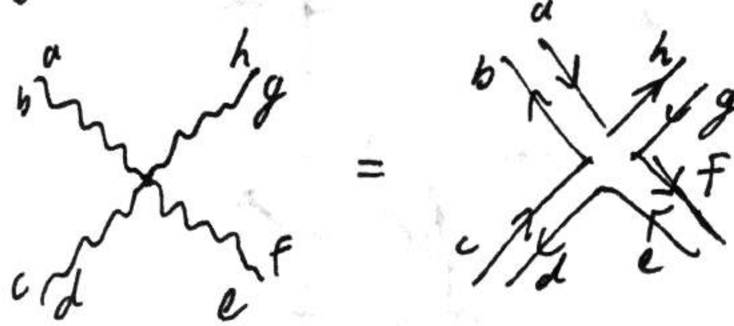
(b) $\propto g^2 N$ (non-planar)

$=$  $+$ 

(a) (b)

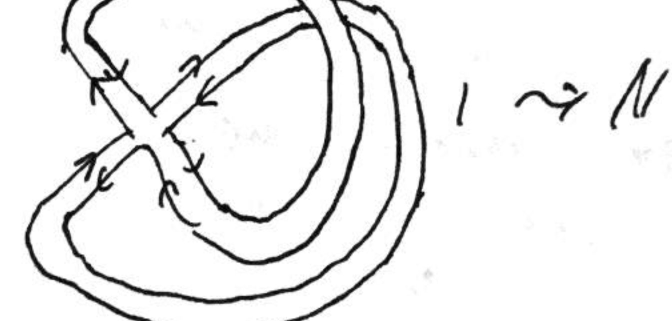
Trick (introduced by 't Hooft): double line notation





(i) a single line : index line
indices connected by a single line are contracted

(ii) direction: from upper to lower indices



Each index loop gives $\sum_a \delta^a_a = N$

another example

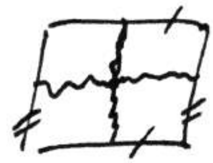
(c)

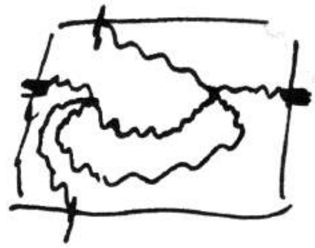
(d)

Empirical evidence:
non-planar diagrams
have smaller
N-dependence

Hints:

(1') These non-planar diagrams can be drawn on the torus without crossing lines

e.g. (b) can be drawn as: 

(d) can be drawn as: 

(2') The power of N for each diagram is equal to the number of faces after we straighten it.

Face: A connected subregion bounded by propagators in a diagram

For planar diagrams, the outside region also counts as a face.

double line: each face is bounded by an index loop

recall: an orientable closed 2-d surface is classified by its number of handles (genus) h

a genus-h surface $\Leftrightarrow$ polygon of $4h$ sides with opposite pairs identified

Generalize 1' and 2' to:

1. For any non-planar diagram, $\exists h$ s.t. the diagram can be drawn on a genus h surface but not lower.
   $\Rightarrow$ diagrams can be classified by genus h
2. N-dependence is given by # of faces on genus h surface

For a general diagram:

$$A \propto \underbrace{(g^2)^E}_{\substack{\#\text{ of}\\ \text{propogators}}} \underbrace{(g^{-2})^V}_{\substack{\#\text{ of interaction}\\ \text{vertices}}} \underbrace{(N)^F}_{\substack{\#\text{ of}\\ \text{faces}}} \quad (*)$$

→ note this is unbounded ~~...~~

⇒ (naively) there is no sensible large N limit

However, note that the Feynman diagrams triangulate the surface. For any triangulation, the Euler characteristic is an invariant of the surface:

$$\chi = F+V-E = 2-2h$$

$$\Rightarrow A \propto (g^2)^{E-V} N^{F+V-E} E^{E-V}$$

$$= (g^2N)^{E-V} N^{2-2h}$$

Now let $g^2 \to 0$, $N \to \infty$

but $\lambda \equiv g^2 N$ finite

$$\underbrace{E-V}_{\substack{\text{number}\\ \text{of indep.}\\ \text{momenta}}} = \underbrace{L}_{\#\text{ of Loops}} - 1 \Rightarrow A \propto \lambda^{L-1} N^{2-2h}$$

sum over genus-h diagrams

$$\Rightarrow (\text{'t Hooft limit}) \quad \log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

$$f_h(\lambda) = \sum_{L=1}^{\infty} f_{hL} \lambda^{L-1}$$

In the $N \to \infty$ limit, Planar diagrams dominate

<u>N.B.</u> Though for general diagrams, the number of contributing diagrams increase <u>factorially</u> in N, for <u>planar</u> diagrams, this instead grows <u>polynomially</u>

We see $\log Z \propto N^2$, and from the Lagrangian we can see this too:

$$\mathcal{L} = -\frac{1}{g^2} \mathrm{Tr}\left(\tfrac{1}{2}(\partial\varphi)^2 + \cdots\right) = -\frac{N}{\lambda} \mathrm{Tr}(\cdots) \sim \mathcal{O}(N^2)$$

2) This discussion only depends on structure, not on detailed ~~information~~ form of $\mathcal{L}$ or its fields (spinor, vector, etc..)

$$\mathcal{L} = N\, \mathrm{Tr}(\cdots)$$

Last time:

$\mathcal{L} = N\,\mathrm{Tr}(\cdots)$ $\quad U(N)$ symmetry

$$\Rightarrow \boxed{\log Z = \sum_{n=0}^{\infty} N^{2-2h} f_h(\{\lambda_\alpha\})}$$

. Correlation Functions

Will restrict our discussion to singlet operators i.e. operators invariant under $U(N)$ symmetry

$\rightsquigarrow$ Such an operator must involve traces

$\mathrm{Tr}\,\Phi^2$, $\mathrm{Tr}\,\Phi^4$, $\mathrm{Tr}(\partial_\mu \Phi \partial^\mu \Phi)$

single-trace $\quad \mathrm{Tr}\,\Phi^2\,\mathrm{Tr}\,\Phi^4 \ldots$

multi-trace

suppose $\{O_k\}_{k=1,2,\ldots}$ denote the set of all single-trace operators, then general singlet operators can be generated from them

$\Rightarrow$ enough to restrict to correlation functions of single-trace operators.

$\langle O_1(x_1) \cdots O_n(x_N)\rangle_c \rightsquigarrow$ c = "connected"

What is the leading order $1/N$ expansion?

There is a simple trick:

fixed external function

$$Z[\{J_i\}] = \int D\Phi \, \exp\left[iS_0 + iN\int J_i \, \mathcal{O}_i(x)\right]$$

$$= \int D\Phi \, e^{iS_{eff}}, \quad S_{eff} = S_0 + N\int J_i(x)\,\mathcal{O}_i(x)$$

$$\Downarrow$$

$$\underline{S_{eff} = N\,\mathrm{Tr}(\ldots)}$$

$$G_n = \frac{1}{i^n}\frac{1}{N^n}\frac{\delta^n \log Z}{\delta J_1(x_1)\cdots\delta J_n(x_n)}\bigg|_{J_i=0}$$

$$\Rightarrow \log Z[\{J_i\}] = \sum_{h=0}^{\infty} N^{2-2h} \cdot F_h(\{J_i, \lambda_a\})$$

$$\Rightarrow G_n = \sum_{h=0}^{\infty} N^{2-2h-n}\, G_n^{(h)}(\{x_i\})$$

without N prefactor

contribution from genus h diagrams

As $N\to\infty$, at leading order

$\langle 1\rangle \sim O(N^2)$

$\langle \mathcal{O}\rangle \sim O(N)$

$\langle \mathcal{O}_1 \mathcal{O}_2\rangle \sim O(1)$

$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3\rangle_c \sim O(N^{-1})$

$\vdots$

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle_c \sim O(N^{2-n})$

Remarks:

1) $\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \frac{1}{4}\Phi^4\right]$
contains other observables which are not singlets under $U(N)$
e.g. $\Phi^d_b\,\Phi^c_d(x)$.
In general, such operators do not have nice scaling with N $\Rightarrow$:(

2) For YM theory, $O(N)$ symmetry is <u>local</u>. Only singlets are allowed. $\Rightarrow$:)

3) Almost <u>all</u> theories of interest to us are gauge theories $\Rightarrow$:D

4) For gauge theories, there are also nonlocal singlet operators such as Wilson loops:

$$W(C) = \mathrm{Tr}\Big(\underset{\uparrow \text{path-ordering}}{P}\exp\big(i\oint_C A\big)\Big)$$

They have the <u>same large-N scaling</u> of single-trace local operators

. The physical nature of the $N\to\infty$ limit

(a) $\langle O_i\rangle \sim O(N) \neq 0$

Variance of O: $\sigma_O^2 = \underbrace{\langle (O-\bar{O})^2\rangle}_{\hookrightarrow O(1)} = \langle O^2\rangle - \langle O\rangle^2$ ← disconnected cancels with this

we include connected and disconnected

$$\Rightarrow \frac{\sigma_O^2}{\langle O\rangle} \sim \frac{1}{N} \to 0 \quad \text{as } N\to\infty$$

i.e. no fluctuations.

similarly, n-point functions factorize

$$\langle O_1 \cdots O_n\rangle = \langle O_1\rangle \cdots \langle O_n\rangle + \cdots$$

is dominated by product of 1-pt functions
"classical"

(b) Redefine $\mathcal{O} \to \mathcal{O} - \bar{\mathcal{O}}$

$\Rightarrow \langle \mathcal{O} \rangle = 0$

Then to leading order in large N

$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \mathcal{O}(1)$

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \cdots +$ all "contractions" $\sim \mathcal{O}(1)$

"Gaussian Theory"

is a "Generalized Free Field Theory"

$\mathcal{O}_i(t, \vec{x})$, but <u>no</u> com connecting $\mathcal{O}_i(t_1, \vec{x})$ with $\mathcal{O}_i(t_2, \vec{x})$

(c) Consider <u>any</u> connected part of the correlation functions.

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_c \sim \mathcal{O}(N^{2-n})$

This is like a tree-level theory of interacting "particles" with coupling $1/N$

Imagine $\mathcal{O}(x)|0\rangle$ "create a single particle"

$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle$ "two-particle"

$\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle$ "n-particle"

 $\sim \frac{1}{N} \sim g$,  $\sim g^2 \sim \frac{1}{N^2}$

$\langle \mathcal{O}_i \mathcal{O}_j \rangle \sim \mathcal{O}(1)$

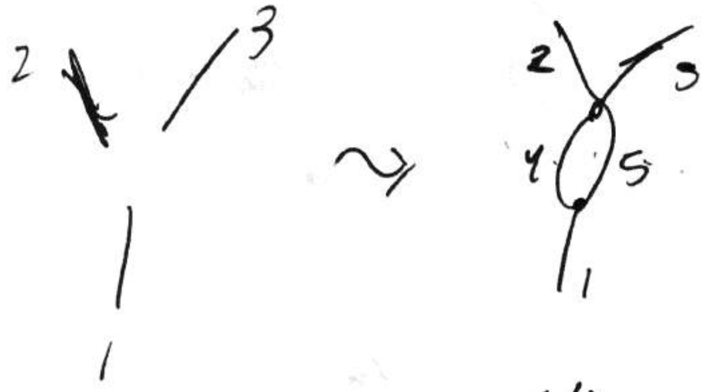
$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim N^{2-n} \sim g^{n-2}$ "tree-level" n-particle scattering amplitude

(d) Adding loop of $\mathcal{O}$'s:

Add intermediate states with more than one $\mathcal{O}$'s

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \langle \mathcal{O}_1 \mathcal{O}_4 \mathcal{O}_5 \rangle \langle \mathcal{O}_4 \mathcal{O}_5 \mathcal{O}_2 \mathcal{O}_3 \rangle - N^{-1} N^{-2}$$

$$\sim N^{-3}$$



Adding loop ⇒ adding $1/N^2$

subleading terms in $1/N^2$: "loop" corrections

2.3 Strings and Matrices

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

$$G_n = \sum_{h=0}^{\infty} N^{2-n-2h} G_n^{(h)}$$

scattering of strings ⟷ Corr. func of single-trace operators

Topology of w.s. ⟷ Topology of Feynman diagrams

g_s ⟷ $1/N$

$A^{(h)}_{n}$: integrate over WS of genus h $\leftrightarrow$ $G_n^{(h)}$ sum over genus h Feynman diagrams

external string $\leftrightarrow$ single-trace $\mathcal{O}$

loops of string $\leftrightarrow$ "loops" of single-trace op

Rough argument:

$$A_0^{(h)} = \sum_{\substack{\text{sum}\\ \text{over genus}\\ h \text{ surfaces}}} e^{iS_{\text{string}}} = \sum_{\substack{\text{triangulations}\\ \text{of genus } h\\ \text{surfaces}}} e^{iS_{\text{string}}}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$G_0^{(h)} = \sum_{\substack{\text{genus-}h\\ \text{Feynman}\\ \text{diagram}}} \tilde{G} = \sum_{\substack{\text{triangulations}\\ \text{of genus } h\\ \text{surfaces}}} \tilde{G}$$

$W(C)$ 

$\langle W(C_1)\, W(C_2)\rangle \simeq$ 

$\cancel{W(C)}$

$A^a_b = {}_a\!\sim_b$

$W(C) = e^{i\oint_c A} \to \bigcirc$

$\Phi^a_b \Rightarrow {}_a\!-\!_b$

$\mathrm{Tr}\,\Phi^2 \Rightarrow {}_a\bigcirc_b$

$\Rightarrow$ $W(C)|0\rangle$ can be considered as <u>macroscopic</u> string states

$\boxed{76}$ $\to$ $\mathcal{O}(x)|0\rangle$: microscopic string state

Oct 24, 2018 (Notes from Sam Leutheusser)

$A_h^{(\lambda)}$: sum over random triangulations of a genus-h surface (and its embeddings) weighted by $e^{iS_{string}}$

$G_h^{(\lambda)}$: sum over triangulations of genus h surfaces weighted by $\mathcal{E}_c$

<u>Conjecture:</u>

a duality of:

a large-N matrix-like theory $\longleftrightarrow$ a string theory

$1/N \longleftrightarrow g_s$

single trace operator $\longleftrightarrow$ single string

To establish this duality:

$S_{string}[\gamma_{ab}, X^\mu, \ldots]$, $X^\mu(\sigma,\tau)$: Worldsheet $\to M$ "spacetime manifold"

$\rightsquigarrow$ Continuum string picture should emerge in regime where "infinitely complicated" Feynman diagrams dominate

<u>Remarks:</u>

(1) So far, only matrix-valued fields have been considered. This also can include fields transforming in the fundamental representation of $U(N)$ $q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$ "quarks"

$\langle q^a q_b \rangle =$ 

Precise mapping to a string theory with both closed and open strings (i.e. quarks add open strings)

(2) In addition to $U(N)$, can also consider $SO(N)$, $Sp(N)$

$\langle \Phi_{ab} \Phi_{cd} \rangle =$ 

(no arrows)

$\Rightarrow$ include non-orientable surfaces

$\rightsquigarrow$ maps to non-orientable string theory

<u>Explicit example:</u> (0-dimensional)

$$e^{-Z} = \int dM \exp\left[-\frac{N}{g} \mathrm{Tr}[V(M)]\right]$$

M = hermitian matrix

$$V(M) = \frac{1}{2}M^2 + \sum_{k=3} a_k M^k$$

$$\rightsquigarrow Z = Z_0 + Z_1 + \cdots$$

$$Z_0 \sim O(N^2)$$

$$dM = \prod_{a,b} dM^a{}_b$$ (for $a=b$ this is the usual $dM^a{}_b \in \mathbb{R}$; for off-diagonal $dM^a{}_b = dM^a{}_b \, dM^a{}_b$)

Since $Tr(V(M))$ depends only on eigenvalues,
write $M = U^{+} \Lambda U$, $\Lambda = \text{diag}(\lambda_1 \dots \lambda_N)$

$\rightsquigarrow$ Measure might depend on λ_i

$\Rightarrow Tr(V(M)) = \sum_{i=1}^{N} V(\lambda_i)$, $dM = \prod_{i=1}^{N} d\lambda_i \, \Delta^2(\Lambda)$ "DU"

It turns out $\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$ (Vandermonde determinant)

shown in problem set

$$\Rightarrow e^{-Z} = \int \prod_i d\lambda_i \, \Delta^2(\Lambda) \, e^{-\frac{N}{g} \sum_i V(\lambda_i)}$$ ← N-sum ⇒ $O(N^2)$

$\rightsquigarrow$ Naive saddle point: $V'(\lambda) = 0$ (incorrect)

$$\Delta^2(\Lambda) = \exp\Big[\underbrace{\sum_{i<j}}_{\substack{\text{double sum} \\ \Rightarrow O(N^2) \text{ too}}} \log(\lambda_i - \lambda_j)^2\Big]$$ ← cant have $\lambda_i = \lambda_j$ as $\log \to -\infty$
"level repulsion"

$\Delta^2(\Lambda)$: 1) $O(N^2)$
2) Repulsion between λ_i s

$\Rightarrow \lambda_i$ cannot all sit at minimum

EOM for λ_i:

$$2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}=\frac{N}{g}V'(\lambda_i) \quad (*)$$

(get this by directly differentiating)

$N\to\infty$ limit (expect $\rho(\lambda)$ to form a continuous function)

$$\int_{-\infty}^{\infty}\rho(\lambda)\,d\lambda=1, \quad \rho(\lambda)=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$$

$$\Rightarrow \boxed{2P\int d\lambda'\,\frac{\rho(\lambda')}{\lambda-\lambda'}=\frac{1}{g}V'(\lambda)} \quad (**)$$

principal value (i.e. throw out $i=j$)

Note:

λ's pushed out, but not to ∞ as this would cause infinite energy so as $N\to\infty$, we get continuous distribution of λ_i (infinite range)

Now assume $\rho(\lambda)$ only supported on finite interval in $\mathbb{R}$, denoted by I

Introduce:

$$F(\xi)=\int_I d\lambda'\,\frac{\rho(\lambda')}{\xi-\lambda'} \quad \text{for general complex } \xi$$

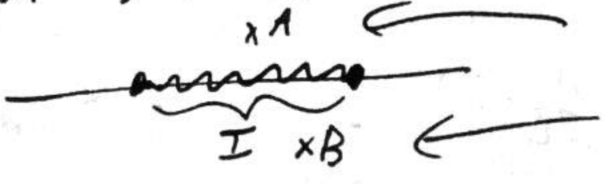
Then $(**)$ is equivalent to: $P\,F(\xi)=\frac{1}{2g}V'(\lambda)$ at $\xi=\lambda-i\varepsilon$ for $\lambda\in I$

The preceding equation comes from the identity:

$$\frac{1}{x \pm i\epsilon - \lambda'} = P\frac{1}{x-\lambda'} \mp i\pi\delta(x-\lambda') \qquad (***)$$

$F(\xi)$ satisfies the following analyticity properties:

(1) Analytic ~~properties~~ function on complex plane except for branch cut at I



compare A and B values using (***) to see branch cut

(2) On real axis, $F(\xi)$ is real for $x \notin I$

(3) $F(\xi) = \frac{1}{\xi} + \dots$ as $\xi \to \infty$

since for $\xi \notin I$, $\frac{1}{\xi - x'} \to \frac{1}{\xi}\frac{1}{1-x'/\xi}$, $\int \rho = 1$

(4) $Re(F(x-i\epsilon)) = \frac{1}{2g}V'(x)$

$Im(F(x-i\epsilon)) = \pi\rho(x)$

These conditions determine $F(\xi)$ (and therefore $\rho(\lambda)$) completely

<u>Example</u>: $V(\lambda) = \frac{\lambda^2}{2} + \lambda^4$,

min at 0

take $I = [-a, a]$, a to be determined

write ansatz for $F(\xi)$:

$$F(\xi) = \frac{1}{2g} V'(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$$

, f is TBD

← pure imaginary for $\xi \in I$

(3) + analyticity gives $f(\xi) = -\frac{1}{2g}(1 + 4x^2 + 2a^2)$

$$a^2 = \frac{1}{6}\left(\sqrt{1+48g} - 1\right)$$

$$\Rightarrow \rho(\lambda) = \frac{1}{2\pi g}\left(4\lambda^2 + 1 + 2a^2\right)\sqrt{a^2 - \lambda^2} \quad \leftarrow \lambda \in [-a, a]$$

$$\Rightarrow Z_0 = N^2\left[-\frac{1}{g}\int_{-a}^{a} d\lambda\, \rho(\lambda) V(\lambda) - P\int_{-a}^{a} d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\log|\lambda - \mu|\right]$$

from potential

from Vandermonde

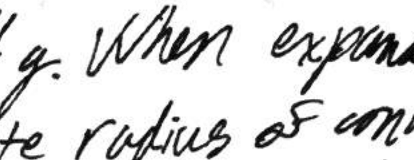
Maps to string theory in 0 dimensions!

Oct 29 (Notes from Sam Leutheusser)

Remarks

(0) $g \to 0$, find $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$ ← Wigner distribution for M gaussian

(1) $Z_0[g]$ is analytic in g for small g. When expanded as a power series, there is a finite radius of convergence

⇝ If g flips sign, one would expect 

⇝ essential sing. at $g=0$ (theory is unstable)

(comes from total # of feynman diagrams $\sim n!$)

But, planar diagrams are polynomial and contribute most at large N

(2) a^2 has branch point in g at $g = g_c = -1/48$ ⇒ radius of conv. $= 1/48$

Perturbation theory breaks down there

near g_c, $Z_0[g] \sim$ analytic in g $+ \alpha (g_c - g)^{5/2} \ldots$ ← planar diagram singularity

(3) From perspective of summing planar diagrams, to see this non analytic behavior one must sum full series (all powers of g)

$$(g_c - g)^{5/2} \sim \sum_n \underbrace{n^{-7/2}}_{\text{non-analytic behavior}} \left(\frac{g}{g_c}\right)^n \leftarrow \text{at large } n$$

(4) Only in $g \to g_c$ limit can we expect a continuum description of string theory to emerge (need arbitrarily complicated triangulations for continuum)

(5) Consistency check: all Z_h $\forall h$ need non-analytic behavior at $g = g_c$ for continuum limit to be string theory

$Z_h[g]$ non-analytic at $g = g_c = -1/48$ $\forall h$ ✓

near g_c, $Z_n[g] \sim |g-g_c|^{\frac{\chi}{2}(2-\Gamma)}$ $\quad$ $\chi = 2-2g$ (Euler char.)
$\Gamma = -1/2$ in this case

(6) String theory dual

$X: (\sigma,\tau) \to M$, γ_{ab}, φ: internal worldsheet d.o.f.

0-d string theory: M is a point, X is trivial

$$Z_{string} = \int D\gamma_{ab}\, D\Phi\, e^{-S_{string}[\gamma,\Phi]}$$ ← spacetime is a point, so work in Euclidean picture

$$S_{string}[\gamma,\varphi] = \mu \underbrace{\int d^2\sigma \sqrt{\gamma}}_{\text{Area, } A} + \underbrace{\lambda \frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma} R}_{\text{Euler number, } \chi = 2-2h} + S_{matter}[\Phi]$$

$$\Rightarrow S_{string} = \mu A + \lambda \chi + S_m[\Phi]$$

Identify $\frac{1}{N} \sim e^{\lambda}$ (controls expansion in genus)

$\underbrace{\mu \propto (g-g_c)^{\#}}_{\text{only param}}$ ← power ends up being 1 since μ is chemical potential for area: $\frac{\partial}{\partial\mu} Z = \langle A \rangle$

so $g \to g_c \Rightarrow n$ in expansion of $(g-g_c)^{n}$ is $\simeq$ area

$$\rightsquigarrow Z_{string} = \sum_h \int D\gamma\, D\Phi\, e^{-\mu A - \lambda\chi - S[\Phi]}$$

After gauge fixing, reparam. invariance of γ allows us to take φ to be the remaining d.o.f. of γ

$$\Rightarrow Z_{string} = \sum_h \int D\varphi\, D\Phi\, \exp[-S_L(\varphi) - S_{matter}(\Phi)]$$

Here $S_L = \int d^2\sigma \underbrace{\sqrt{\gamma_0}}_{\text{gauge fixed}} \left[(\partial\varphi)^2 + \varphi \cdot \underset{\uparrow\ \text{parameter}}{Q} R + \mu e^{2\gamma\varphi} \right]$

$\Rightarrow$ φ behaves like an inhomogeneous "emergent" spatial dimension. Even though X^μ is trivial, we grow a dimension.

$$Z_h^{(\text{string})} \propto \mu^{\frac{\chi}{2}(2-\Gamma_{\text{string}})}$$

For S_{matter} trivial, we find $\Gamma_{\text{string}} = -1/2$.

(matches matrix theory)

2.3 String Description of a ~~ga~~ Gauge theory

Non-Abelian $SU(N)$ gauge theory $N\to\infty$ in d-dim $\Leftrightarrow$ String theory?

A simplest guess: Maybe a string theory in $\underbrace{\text{Mink}_d}_{ds^2 = -dt^2 + d\vec{x}^2}$

$\rightsquigarrow$ This does not work!

(1) Such a string theory appears inconsistent
A string theory in Mink_d is consistent only for $d=26$ (bosonic), $d=10$ (superstring)

(2) ~~A~~ How about $\text{Mink}_4 \times N$ — compact
so string theory has $SO(3,1)$ symmetry

But all string theories on $\text{Mink}_4 \times N$ have 4d graviton $\to$ violates Weinberg-Witten

W-W applies because $SU(N)$ theory has gauge-inv conserved stress tensor.

→ We'd want a string theory "without gravity"?
→ impossible!

(Hints:

(1) φ with Mink_4 combines to form 5d curved spacetime
→ 5d non-compact spacetime
→ 5d gravity
→ Does not contradict Weinberg-Witten

(2) Holographic Principle
4d gauge theory could in principle be related to 5-d gravity
→ One could try a string theory in $Y_5 \times M$

non-compact spacetime

compact space

$\Rightarrow Y_5$ should have all the Mink_4 symmetries

$$\rightsquigarrow ds^2 = g(z)\left(-dt^2 + dx^2\right) + \underset{f(z)}{\underbrace{\;}} dz^2$$

$$= \Omega^2(z)\left(-dt^2 + dx^2 + dz^2\right) \quad (*) \text{ after redefinition}$$

Suppose the 4-d theory is scale-invariant

$$(**) \quad (t, \vec{x}) \to \lambda (t, \vec{x})$$

(*) must be invariant under (**)

$\Rightarrow$ must have $z \to \lambda z$, ~~$\Omega(z) \to \Omega(\lambda z)$~~

$$\Omega(\lambda z) = \frac{1}{\lambda}\Omega(z)$$

This fixes $\Omega(z) = c/z$

So, for scale-invariant lower dimensional theory, we would have

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + d\vec{x}^2 + dz^2\right]$$

R is some constant

This is AdS.

AdS_5 has isometry group $SO(2,4)$

881

Last time:

$\underbrace{Y_5}_{\text{non-compact}} \times \underbrace{\mathcal{N}}$ does not contradict Weinberg-Witten

$$ds^2 = \Omega^2(z)\left(-dt^2 + dx^2 + dz^2\right)$$

← 1997 Polyakov wrote this down to study QCD.
Nov. 1997 Maldacena wrote this, realizing AdS

scale-invariant: $(t, \vec{x}) \to \lambda (t, \vec{x})$
$z \to \lambda z \quad \Rightarrow \Omega(\lambda z) = \frac{1}{\lambda}\Omega(z)$

$\Rightarrow \Omega(z) = \frac{R}{z}$ (R: const)

$$\Rightarrow \underbrace{ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right)}_{AdS_5 \text{ space}}$$

determined uniquely (up to R)

scale invariance $\rightsquigarrow$ conformal invariance $\underbrace{SO(2,4)}_{\text{isometry}}$

- Outline of string theory & derivation of AdS/CFT

(a) closed strings

quantization of a closed string in a fixed spacetime:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X_\mu \quad (*)$$

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau)$$

Tricky to quantize, but the procedure is well-known

⇒ String ~~theory~~ excitation spectrum

(1) Not all spacetime allow a consistent string propagation quantum mechanically

~~(2)~~ For (*), we have bosonic string theory:

<u>Mink</u>: $d=26$

Taking (X^μ, ψ^μ) : superstring
super coordinates

Mink: $d=10$

~~(3)~~ (2) Spectrum:
Oscillation excitation of a string ↔ spacetime particle

<u>Massless</u>: $\underbrace{h_{\mu\nu}, B_{\mu\nu}, \Phi}_{\text{universal to all string theories}}, \ldots$

<u>Massive</u>: $m^2 = N/\alpha'$
(infinite towers of massive modes)
$\alpha' = l_s^2$ l_s: string length

$h_{\mu\nu}$: massless spin-2 (graviton)

$\underbrace{B_{\mu\nu}}_{\text{antisymmetric tensor}} = -B_{\nu\mu}$:

↔ $B_{\mu\nu}$ "gauge field" for a string ($U(1)$)

Φ : $g_s = \langle e^\Phi \rangle$ ← g_s can vary spatially if Φ does / coupling of the string
g_s ← "free parameter"

$G_N \propto g_s^2 \Rightarrow G_N = 8 g_s^2 \alpha'^{\frac{d-2}{2}}$

At low energies: $E^2 \ll 1/\alpha'$

effective theory: Einstein gravity + matter (massless)

+ higher derivative corrections

Due to presence of graviton:

spacetime metric becomes dynamical

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$\Rightarrow$ closed strings must themselves be excitations of spacetime

(* this logic is still not clear to me)

(3) Bosonic string:

there exists excitation with negative N

i.e. $m^2 < 0$

$\Rightarrow$ unstable

Superstrings can be stable

open question

5 different perturbative superstrings

We now know they are related nonperturbatively

Their spectra contain spacetime fermions

At low energies: Supergravity

(interesting note: there are "other" superstring theories with only bosons but emergent fermions.)

Massless: $(d=10)$

II A: $h, B^{(2)}, \Phi, C_\mu, C^{(3)}_{\mu\nu\lambda}$, + fermions

II B: $h, B^{(2)}, \Phi, \chi, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\nu\lambda\rho}$ + fermions

$C^{(2)}, C^{(3)}, C^{(4)}$ are fully anti-symmetric

$C^{(4)}$: self-dual

$F^{(5)} := dC^{(4)}, \quad F^{(5)} = * F^{(5)}$

(b) Open string

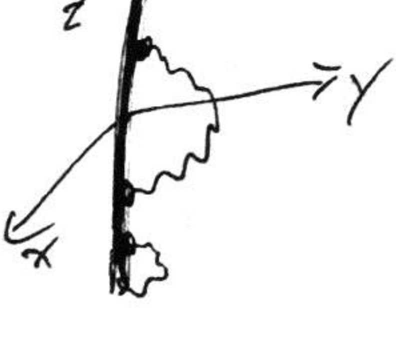
need to impose boundary conditions at ~~the~~ end points

They can only end on some "special places" which can be considered as some kind of "defects" in spacetime

"D-branes"

a D_p-brane has p spatial dimensions

D1-brane



D3-brane "spacetime-filling"

$\rightsquigarrow$ can end anywhere

Classically, D-brane can be considered as a proxy of specifying boundary conditions on an open string

<u>D0-brane</u>

Given a D-brane configuration, you can quantize open string ending on it.

quantize open strings:

<u>massless</u>: $A_\mu(x^\nu)$

$\mu, \nu = 0, \dots, p$

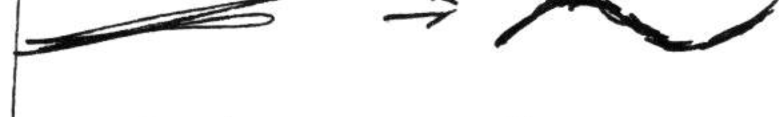
$\Phi^i(x^m)$ $i = p+1, \dots d-1$

<u>massive</u>: $m^2 = N/\alpha'$

endpoint of a string: charged particle

A_μ: gauge Field for the charge

Φ^i: transverse motion of the brane <u>itself</u>



$\Rightarrow$ D-branes are dynamical objects

$\Rightarrow$ open strings are excitations of D-branes

Some D-branes have $N<0$ $\Rightarrow$ unstable excitations

$\rightsquigarrow$ decay into closed string modes

There are stable-situations

IIA 0, 2, 4, 6, (8) (even-dimensional)

IIB (-1), 1, 3, 5, 7, (9) (odd-dimensional)

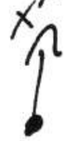
An object is stable if it carries some sort of "gauge" charge*.

A p-dimensional object can carry charge of $(p+1)$-form

Nov 5, 2018

$\sigma \in [0, \pi]$

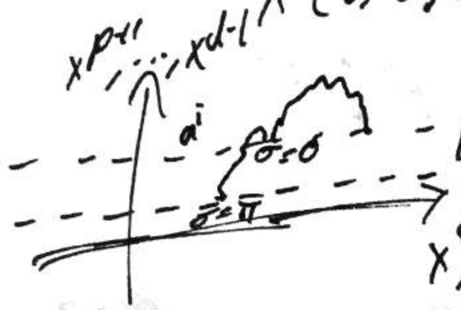
$$X^\mu(\sigma, \tau) \rightsquigarrow X^\mu(0,\tau) \Rightarrow \begin{cases} a^\mu & \text{Dirichlet} \\ \partial_\sigma X^\mu\big|_{\sigma=0} = 0 & \text{Newman} \end{cases}$$



"no momentum exits from the string"

$\underline{\sigma = 0}$ $\quad X^\mu(0,\tau) \quad \mu = 0, 1, \cdots p \quad$ Newman

$\quad X^i(0,\tau) = a^i \quad i = p+1, \cdots, d-1 \quad$ Dirichlet



Massless excitations: $\quad A_\mu(x^\nu) \quad \mu, \nu = 0, 1, \cdots p$

$\quad \Phi^i(x^\mu) \quad i = p+1, \cdots, d-1$

Massive: $\quad m^2 = N/\alpha' \quad \leftarrow$ again can be negative $\Rightarrow$ unstable D-branes

$\Rightarrow$ D-branes are dynamical objects

$\Rightarrow$ Open strings are excitations of D-branes

Stable D-branes:

IIA: even dimensions

$\quad 0, 2, 4, 6, (8)$

IIB: odd-dimensional ones

$\quad (-1), 1, 3, 5, 7, (9)$

$h_{\mu\nu},\ B_{\mu\nu},\ \Phi + \begin{cases} C^{(1)}_\mu,\ C^{(3)}_{\mu\nu\lambda} & \text{IIA} \\ \chi,\ C^{(2)}_{\mu\nu},\ C^{(4)}_{\mu\nu\lambda\rho} & \text{IIB} \end{cases}$ Massless closed string spectrum

Anti-symmetric potentials are generalization of $U(1)$ Maxwell field to higher forms

$$A_\mu \qquad A = A_\mu dx^\mu$$

$$F = dA \qquad \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$C^{(n)} = C_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$F^{(n+1)} = dC^{(n)} \qquad \mathcal{L} = -\frac{1}{2}\frac{1}{n!}(F \wedge * F)$$

$$C^{(n)} \to C^{(n)} + d\Lambda^{n-1}$$

$\Rightarrow$ point charged particle with path x^μ couples to A as $\int_C A_\mu \frac{dx^\mu}{d\tau} d\tau = \int_C A$ ← pullback to particle worldline

p-dimensional object has $(p+1)$-world volume Σ

$$\int_\Sigma C^{(p+1)} \leftarrow \text{pullback of } C^{(p+1)} \text{ to } \Sigma \quad = \int_\Sigma d^{p+1}\xi\, C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial \xi^1} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \xi^{p+1}}$$

with $X^\mu(\xi)$ denoting embedding of Σ in spacetime

For any $C^{(n)} \to F^{(n+1)} = dC^{(n)} \to \tilde{F}^{(d-n-1)} = * F^{n+1}$

$$\tilde{F}^{d-n-1} = d\tilde{C}^{(d-n-2)}$$

Given $C^{(n)}$ there is a $p = n-1$ dimensional object charged under "electric"

and a $d-n-3$ dimensional object charged under "magnetic"

$B_{\mu\nu}$: String electric

NS5-brane magnetic

IIA: $C^{(1)}$ D0-brane (E)

D6-brane (M)

$C^{(3)}$ D2-brane (E)

D4-brane (M)

IIB: χ D(-1)-brane (E)

D7-brane (M)

$C^{(2)}$ D-string (E)

D5-brane (M)

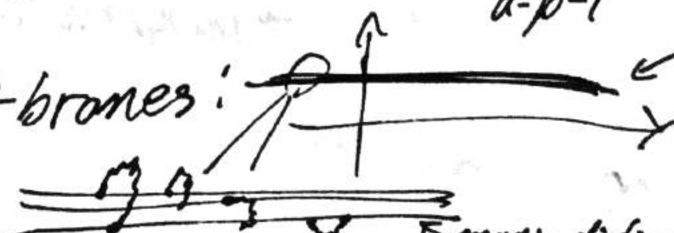
$C_+^{(4)}$ D4 brane (EM)

Properties of D-branes:

- At low energies

A single D-brane $\Rightarrow$ $U(1)$ Maxwell + $\underset{d-p-1}{\Phi^i}$ (free scalar)

N coincidental D-branes:

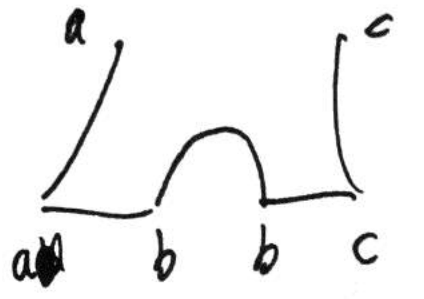


$\rightsquigarrow$ Each string has two endpoints $\Rightarrow$ two labels $1, \ldots, N$

$\Rightarrow$ massless fields $(A_\mu)^a{}_b$ $a, b = 1, \ldots, N$

$(\Phi^i)^a{}_b$ ← you can show these are hermitian

$\Rightarrow U(N)$



$\Rightarrow$ matrix product

low-energy limit:

$$S = -\frac{1}{g_{YM}^2}\int d^{p+1}x \, Tr\left(\frac{1}{4}F^2 + D_\mu \Phi^i D^\mu \Phi_i + [\Phi^i, \Phi^j]^2 + \dots\right)$$

(fermionic terms)

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu] \qquad g_{YM}^2 \sim g_s$$

$$D_\mu \Phi^i = \partial_\mu \Phi^i - i[A_\mu, \Phi^i]$$

$\boxed{\text{Maximally supersymmetric YM}}$

D3-brane: $\Rightarrow p+1 = 4$ $\quad N=4$ SYM

When we separate the branes:

$$\langle \Phi^i \rangle^a_a \neq 0$$

$$U(N) \to U(n_1) \times U(n_2) \dots \times U(n_R)$$

- D-branes gravitate

a charged point particle in $D=4$

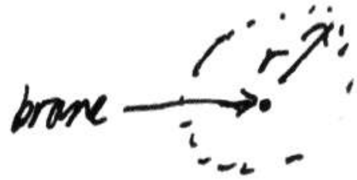
$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G_N} R - \frac{1}{4}F^2\right] - m\int_c ds + q\int A$$

$$\partial_\mu F^{\mu\nu} = j^\nu \text{ (particle)}$$

$$G_{\mu\nu} = 8\pi G_N (T_{\mu\nu}(\text{particle}) + T_{\mu\nu}(EM))$$

E.g. Consider a D3-brane

$\underline{1,2,3} \quad \underline{4,5,6,7,8,9}$



N ~~D~~D3-branes

$$ds^2 = f(r)\left(-dt^2 + d\vec{x}^2\right) + g(r)\left(dr^2 + r^2 d\Omega^2\right)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}, \quad R^4 = 4\pi g_s (\alpha')^2 N \propto N T_3$$

tension of D3-brane

$r = 0 \rightarrow$ location of D3-brane

$r \rightarrow \infty \rightarrow$ "Minkowski space"

R characterizes when gravitational effect becomes strong.

note that Schwarzschild metric has a $-$ sign

At $r \rightarrow 0$, $f(r) = \frac{r^2}{R^2}$, $g(r) = \frac{R^2}{r^2}$

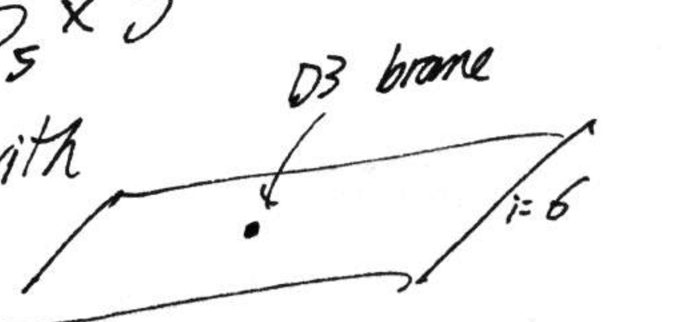
$$\Rightarrow ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right)$$

$$= \underbrace{\frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2} dr^2}_{AdS_5} + R^2 d\Omega_5^2$$

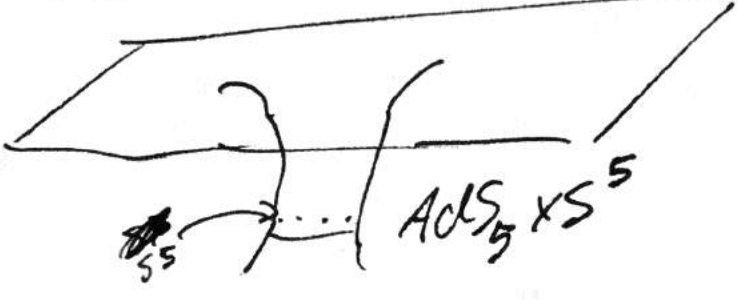
S^5 never shrinks to 0 size (const radius $= R$)

$AdS_5 \times S^5$

Started with



Became:



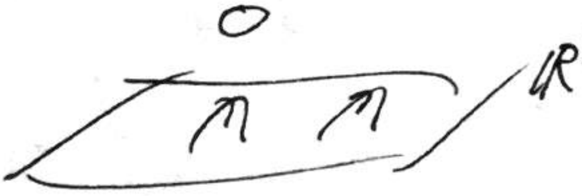
$\frac{dr^2}{r^2} = dl^2$

$l = \log r$

($r = 0$ sits an infinite proper distance away)

We thus have two descriptions of D3-branes

(A) D-branes in flat $Mink_{10}$ where open strings can live

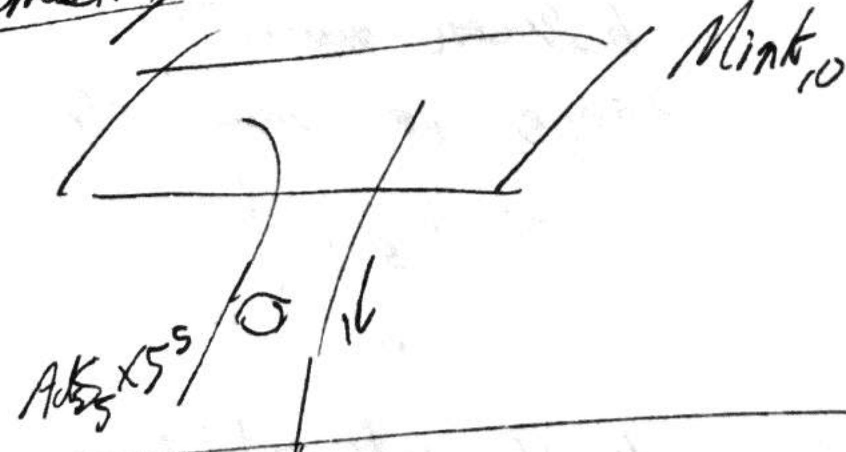


(B) Using spacetime metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

(+ F_5 flux on S^5)

$\rightsquigarrow$ only <u>closed strings</u> that see a <u>curved geometry</u>



$$A = B$$

Both descriptions can in principle be valid for all α' and g_s

Maldacena (1997)

low energy limit $\Rightarrow$ AdS/CFT

What is the low energy limit?

Fix E, take $\alpha' \to 0$
(Fix α', $E \to 0$) $\rightsquigarrow \alpha' E^2 \to 0$

(A): Open string sector $\Rightarrow$ $\mathcal{N}=4$ Super Yang-Mills with gauge group $U(N)$

$$g_{YM}^2 = 4\pi g_s$$

Closed string sector $\Rightarrow$ graviton, dilaton

couplings between massless closed and open strings, or closed strings themselves

$$G_N \propto g_s^2 \alpha'^4$$

$$E \to 0 \Rightarrow G_N E^8 \to 0$$

So we get the interacting $\mathcal{N}=4$ theory + free massless modes as $E \to 0$

(B): As before:

$$ds^2 = f(r)(-dt^2 + d\vec{x}^2) + g(r)(dr^2 + r^2 d\Omega_5^2)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}$$

Curved spacetime: must be careful to specify what "energy" to use

E in (A): defined w.r.t. t (i.e. time at $r=\infty$)

At r: local proper time

$$d\tau = g^{-1/2} dt$$

$$\Rightarrow E_\tau = g^{1/2} E$$

For $r \gg R$: $g \sim 1$

$E^2 \alpha' \to 0 \Rightarrow$ all massive closed strings decouple

For $r \ll R$ $\quad g \sim \frac{R^2}{r^2}$

$$E^2 \alpha' \to 0 \Rightarrow E_\tau^2 \frac{r^2}{R^2} \alpha' \to 0$$

$$\Rightarrow E_\tau^2 \frac{r^2}{\sqrt{4\pi g_s N}} \to 0$$

$\Rightarrow$ For any E_τ, low energy limit has $r \to 0$

So Low energy limit gives free gravitons at $r = \infty$ (Mink_{10})

+ full string theory in $AdS_5 \times S^5$ (with flux)

$$A \quad = \quad B$$

$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$ (low energy limit)

$\mathcal{N}=4$ SYM theory + free graviton $\qquad$ IIB String in $AdS_5 \times S^5$ + free graviton

$\Rightarrow$ $\boxed{\text{$\mathcal{N}=4$ SYM theory with gauge group $U(N)$} \quad = \quad \text{IIB String in $AdS_5 \times S^5$}}$

Wednesday Nov. 14, 2018

$\mathcal{N}=4$ SYM theory with gauge group $U(N)$ in $M_4 = (1,3)$

IS

IIB String theory in AdS

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx^2\right) + \frac{R^2 dr^2}{r^2} + R^2 d\Omega_5^2$$

plus F_5^+

$$g_{YM}^2 = 4\pi g_s \rightsquigarrow \lambda = g_{YM}^2 N = \frac{R^4}{(\alpha')^2}$$

$$\underbrace{16\pi G_N}_{\text{fundamental}} = (2\pi)^7 g_s^2 (\alpha')^4 \Rightarrow \frac{G_N}{R^8} = \frac{\pi^4}{2N^2}$$

2.5 Anti-de Sitter spacetime AdS_{d+1}

Homogenous spacetime of constant negative curvature. Consider hyperboloid in $(2,d)$

$$X_{-1}^2 + X_0^2 - \sum_{i=1}^{d} X_i^2 = R^2$$

with metric $ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^{d} dX_i^2$

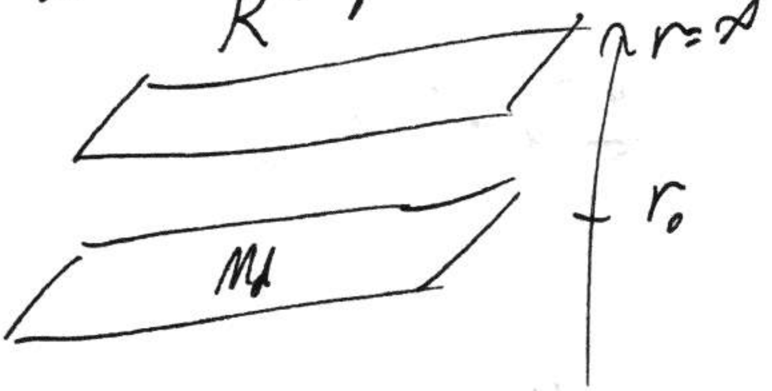
$\rightsquigarrow SO(2,d)$ isometry

(i) <u>Poincare coordinates</u>

$r = X_{-1} + X_d, \qquad x^\mu = R\,\frac{X^\mu}{r} \qquad \mu = 0, \dots, d$

$r > 0$

$$\Rightarrow ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2$$



(ii) <u>Global coordinates</u> take $X_{-1} = R\sqrt{1+\rho^2}\cos t \quad X_0 = R\sqrt{1+\rho^2}\sin t$

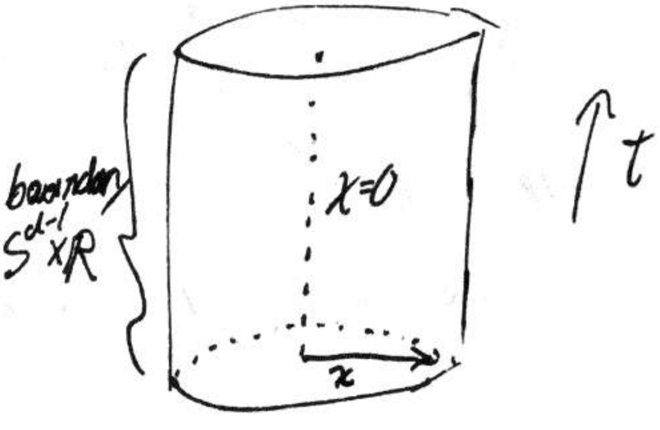
$$\sum_{i=1}^{d} X_i^2 = R^2\rho^2 \rightsquigarrow X_{-1}^2 + X_0^2 = R^2(1+\rho^2)$$

$$\Rightarrow ds^2 = R^2\left[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_{d-1}^2\right]$$

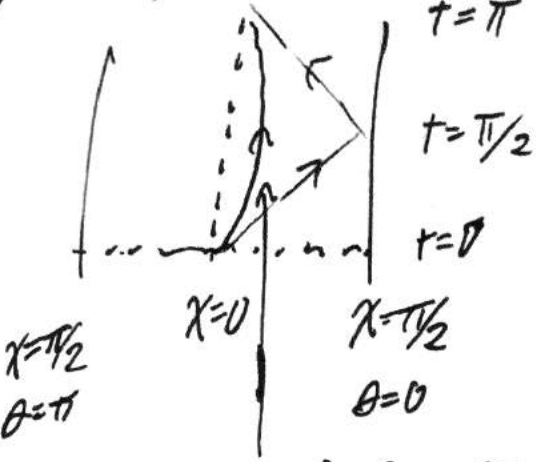
$\rho = \tan\chi \quad (\chi \in [0, \pi/2))$

$$ds^2 = \frac{R^2}{\cos^2\chi}\left[-dt^2 + d\chi^2 + \sin^2\chi\, d\Omega_{d-1}^2\right]$$

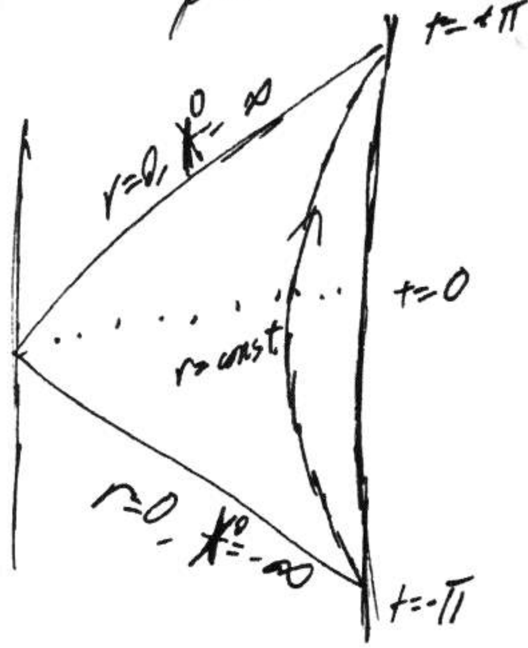
The causal structure of AdS_{d+1} is that of a solid cylinder: $B^d \times \mathbb{R}$

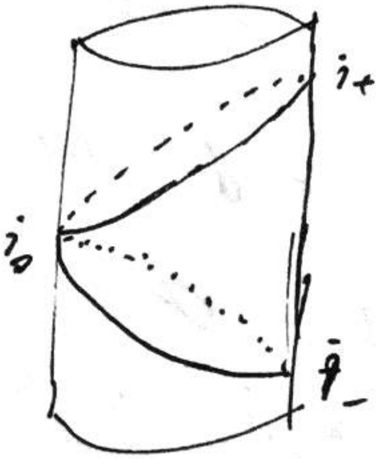


Light ray can reach the boundary in finite time



a massive particle cannot. It will be pulled back by gravitational pull





global AdS contains infinite # of copies of Poincare patch.

· Symmetries of AdS_{d+1}

isometry: $SO(2,d)$, $\frac{1}{2}(d+2)(d+1)$ generators

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 = \frac{R^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \quad (*)$$

p^μ: Translation along x^μ d

$M^{\mu\nu}$: Lorentz trans. for x^μ $\frac{1}{2}d(d-1)$

scaling: $z \to \lambda z$, $x^\mu \to \lambda x^\mu$

(*) is also invariant under

$$I = \begin{array}{l} z \to \dfrac{z}{z^2+x^2} \\ x^\mu \to \dfrac{x^\mu}{z^2+x^2} \end{array} \Bigg\} \text{ not connected to identity}$$

$\leadsto$ special conformal: $I \circ P^\mu(b_\mu) \circ I$ $\quad d$

$$z' = \frac{z}{1+2b\cdot x+b^2(z^2+x^2)}$$

$$x^{\mu\prime} = \frac{x^\mu + b^\mu(x^2+z^2)}{1+2b\cdot x+b^2(z^2+x^2)}$$

$\frac{1}{2}(d+2)(d+1) = d+\frac{1}{2}d(d-1)+1+d$ ✓

. Symmetries of S^n: $SO(n+1)$

$\leadsto$ $AdS_5 \times S^5$: $SO(2,4)\times SO(6)$

String Theory in $AdS_5 \times S^5$

g_s, α', R

$\leadsto$ Two dimensionless parameters

$g_s, \dfrac{\alpha'}{R^2}$ i.e. $\left(\dfrac{G_N}{R^8}, \dfrac{\alpha'}{R^2}\right)$

. classical gravity as $g_s \to 0$, $\dfrac{\alpha'}{R^2} \to 0$ $\quad \alpha'/R^2 = l_s^2/R^2$

string weakly interacting $\quad$ massive modes decouple

$\Rightarrow$ IIB supergravity

= Einstein gravity + finite # of matter fields

• "classical" string limit

$$\frac{\alpha'}{R^2} = \text{finite}, \quad g_s \to 0$$

$$\varphi(x^\mu, z, \Omega_5) = \sum_{\ell} \underbrace{\varphi_\ell(x^\mu, z)}_{\substack{\text{fields in} \\ AdS_5}} \underbrace{Y_\ell(\Omega_5)}_{\substack{\text{harmonics} \\ \text{on } S^5}}$$

$\Rightarrow$ 5-dimensional gravity

$$S_{\text{gravity}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\, R_5 + \text{matter}$$

$$\Rightarrow G_5 = \frac{G_N^{(10)}}{\underbrace{V_5}_{\text{Volume of } S^5}} = \frac{G_N}{\pi^3 R^5}$$

• $\mathcal{N}=4$ SYM (3+1)

Field content: A_μ, φ^i $(i=1,\cdots,6)$, χ^A_α $(A=1,\cdots,4)$

all in adjoint rep'n of $U(N)$

$\sim$ all $N\times N$ hermitian matrices

altogether: $(8b + 8f) \times N^2$

on-shell d.o.f.

Interacting part: $SU(N)$

$U(1)$ part: Free

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \mathrm{Tr}\left(\frac{1}{4} F^2 + \frac{1}{2}(D_\mu \varphi^i)(D^\mu \varphi^i) + [\varphi^i, \varphi^j]^2 + \text{fermionic part}\right)$$

Properties:

(1) Has $N=4$ supersymmetries

Susy: boson $\leftrightarrow$ fermion

$\Rightarrow$ conserved fermionic charges

trans parameter: spinor (Weyl)

4 such indep. spinor parameters

The "simplest strongly-interacting 4-D theory"

(2) g_{YM} is a dimensionless quantity

<u>and</u>

β-function is 0

(3) Conformally-invariant.

Continuing: (Nov 19, 2018)

IIB String theory in $AdS_5 \times S^5$ = $\mathcal{N}=4$ SYM with $U(N)$ in $Mink_4$

$\|$

5-dimensional quantum gravity

$\downarrow$

classical limit

$\|$

$(A_\mu, \phi^i, i=1,\cdots,6, \chi^A_\alpha, A=1,\cdots,4)$

- Maximally supersymmetric theory in $d=4$ (4 susys)
- β-function g_{YM} is zero
  g_{YM}: true dimensionless param
- conformally invariant
  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
  $x^\mu \to x'^\mu(x)$ s.t. $g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$ — overall scaling

For $g_{\mu\nu} = \eta_{\mu\nu}$, such transforms are:

$x'^\mu = x^\mu + a^\mu$

$x'^\mu = \Lambda^\mu_\nu x^\nu$

$x'^\mu = \lambda x^\mu$

$x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$ — "special conformal"

$S = I \cdot T(b) \cdot I$

inversion

translation

$SO(2,d)$ — conformal group:

Poincare: $P^\mu, M^{\mu\nu}$

scaling: D

special conformal: K^μ

(super) Conformal Field Theory

The full bosonic symmetries are

$$SO(2,4) \times \underset{\substack{\uparrow \\ \text{rotate } \varphi^i \text{ (and } \chi^A_\alpha)}}{SO(6)}$$

So with SUSY included, the full (super) group of symmetries is

$$PSU(2,2|4)$$

- Remarks on CFTs

basic objects: <u>local operators</u> with definite scaling dimensions

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^\Delta \mathcal{O}(\lambda x)$$

Δ: dimension

Hilbert space: fall into reps of $SO(2,d)$

typical observables: <u>correlation functions of local operators</u>

conformal symmetry determines the 2 and 3 point correlation functions up to constants:

$$\langle \mathcal{O}_1(x), \mathcal{O}_2(y)\rangle = \frac{C}{|x-y|^{2\Delta_1}}\,\delta_{\Delta_1,\Delta_2}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta \cdots}}$$

$x_{ij} = x_i - x_j$

Remarks:

(1) Isometries of $AdS_5 \times S^5$ form a subgroup of general coordinate transformations, which are local symmetries on gravity side

(2) Isometry: This is the subgroup of coordinate transformations that leave the asymptotic form of the metric invariant "large gauge transformations"

(by analogy: $A_\mu \to A_\mu + \partial_\mu \Lambda(x)$
$\Lambda(x) \to 0$ as $|x| \to \infty$ usually
but if $\Lambda(x) \to$ const. as $|x| \to \infty$
it is "large")

· Match parameters:

gravity:		$\mathcal{N}=4$:
$4\pi g_s$	$=$	g_{YM}^2
$R^4/(\alpha')^2$	$=$	$\lambda \equiv g_{YM}^2 N$
G_5/R^3	$=$	$\pi/2N^2$

classical gravity:

$G_5/R^3 \to 0 \Rightarrow N \to \infty \Leftrightarrow$ strong coupling and large N limit

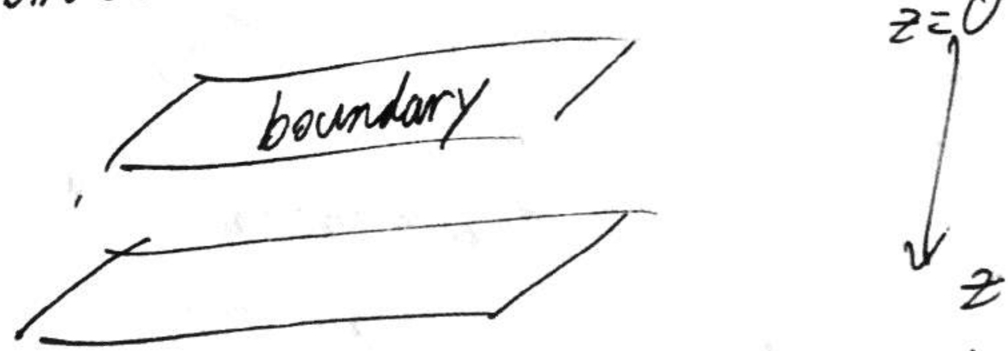
$(\alpha')^2/R^4 \to 0 \Rightarrow \lambda \to \infty$

$1/N^2 \leftrightarrow$ QG corrections

$1/\sqrt{\lambda} \leftrightarrow \frac{\alpha'}{R^2}$ string corrections

- An example of an equivalence between matrices and strings
- Can also be considered as an example of the holographic principle

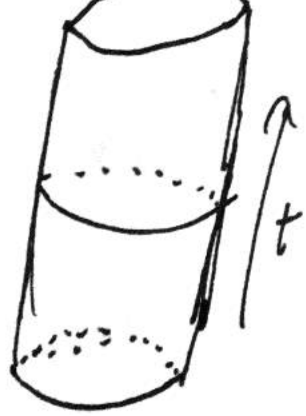
In Poincare coordinates:



$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \quad z \in (0, \infty)$$

boundary of AdS_5 = $Mink_4$

Holographic perspective $\Rightarrow$ prediction:



boundary of global AdS_5 $= S^3 \times \mathbb{R}$

quantum gravity in global AdS_5
$\shortparallel$
$\mathcal{N}=4$ SYM on $S^3 \times \mathbb{R}$

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Chapter 3:
Holographic Duality

$$\text{quantum gravity in } AdS_{d+1} = CFT_d$$

Equivalence between two quantum systems

$\rightsquigarrow$ guess the dictionary

$\Rightarrow$ verify

$\Rightarrow$ make more guesses

- parameters, symmetries <u>should match</u>

e.g. $U(1)$ gauge $\leftrightarrow$ $U(1)$ global

3.1 General aspects

3.1.1. IR/UV connection

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \qquad (*)$$

$(*)$ is invariant under $(t, \vec{x}) \to \lambda(t, \vec{x})$
$z \to \lambda z$

$\Rightarrow$ extra dimension $\leftrightarrow$ scale

Note: $(t, \vec{x})$ defined in boundary units

$$d\tau = \frac{R}{z} dt \qquad dl = \frac{R}{z} dx$$

→ For some process of local energy E_{loc} and local length d_{loc} at z

$$d_{YM} = \frac{z}{R} d_{loc}, \quad E_{YM} = \frac{R}{z} E_{loc}$$

⇒ For the same process at different z

(boundary) $z \to 0$: $E_{YM} \to \infty$, $d_{YM} \to 0$ (UV process)

$z \to \infty$: $E_{YM} \to 0$, $d_{YM} \to \infty$ (IR process)

⇒ typical bulk process, $E_{loc} \sim 1/R$

$$\Rightarrow E_{YM} \sim 1/z$$

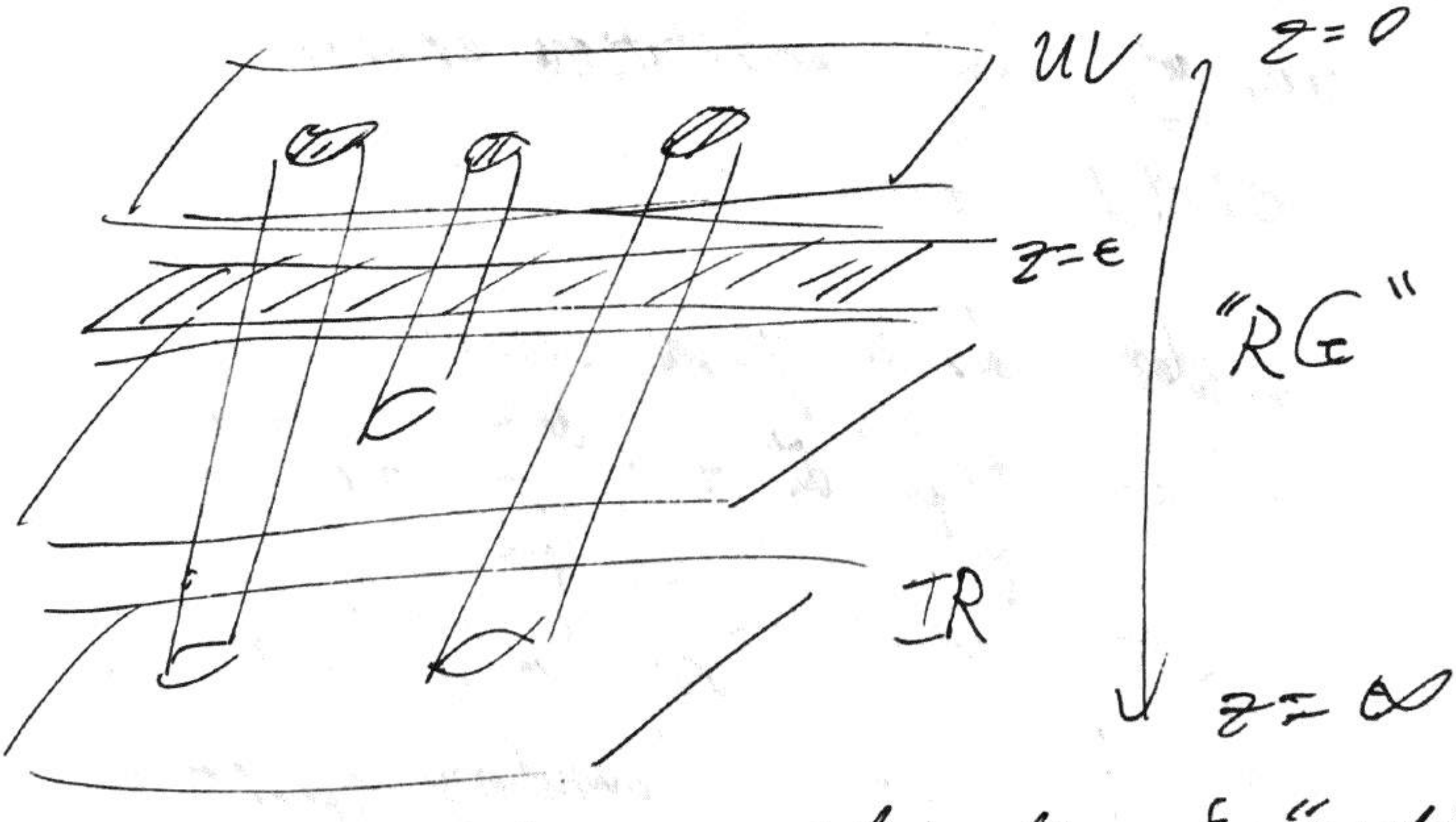
IR-UV connection

Wednesday Nov 21, notes from Haoyo Guo

From before:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right)$$

$$\left.\begin{aligned} E_{YM} &= \frac{R}{z} E_{loc} \\ d_{YM} &= \frac{z}{R} d_{loc} \\ &\cancel{E_{YM} \propto \tfrac{1}{z} \quad d_{YM} \propto z} \\ E_{loc} &\sim 1/R, \quad d_{loc} \sim R \end{aligned}\right\} \Rightarrow \begin{aligned} E_{YM} &\propto \frac{1}{z} \\ d_{YM} &\propto z \end{aligned}$$



radial direction: geometrization of "scale"

IR-UV connection

Remarks:

1) $z \to 0$ $\quad E_{YM} \to \infty$

$\quad\quad\quad\quad d_{YM} \to 0$

put an IR-cutoff on gravity side at $z = \epsilon$

$\Rightarrow$ UV cutoff $\propto 1/\epsilon$ (energy)

distance $\propto \epsilon$

Check: basic idea of holographic principle (pset)

2) In a CFT on $R^{3,1}$ there exist arbitrarily low excitation energies.

reflected in $z \to \infty$ on the gravity side

3) Consider AdS in global coordinates:

$$ds^2 = -\underbrace{\left(1+\frac{r^2}{R^2}\right)}_{f} dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2$$

$S^{d-1} \times t$

on field theory side boundary sphere has radius R

$E_{YM} \sim 1/R$ (energy gap)

$$E_{YM} = f^{1/2} E_{loc}$$
$$= \left(1+\frac{r^2}{R^2}\right)^{1/2} E_{loc}$$
$$= \begin{cases} \infty & \text{as } r \to \infty \\ E_{loc} \sim 1/R & \text{as } r \to 0 \end{cases}$$

4) This works in more general asymptotic AdS metric

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2 d\Omega_{d-1}^2$$

away from boundary f decreases
$\Rightarrow E_{YM}$ decreases

3.1.2 Matching of the spectrum

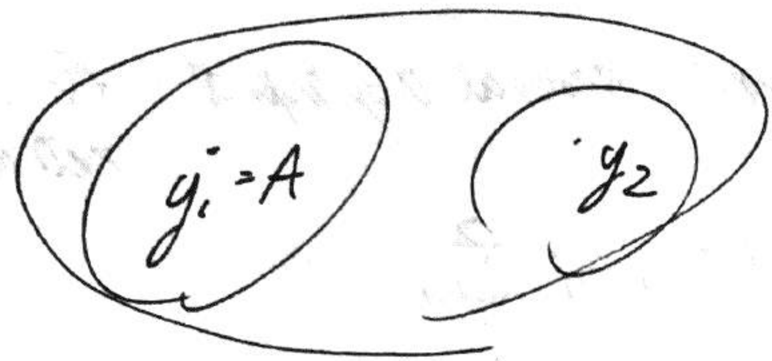
$$\underbrace{QG \text{ in } AdS_{d+1} = CFT_d}_{\text{same Hilbert space}}$$

physical states $\longleftrightarrow$ physical states

Classical gravity $\rightarrow$ Classical solutions (states)

geometry $\not\Rightarrow$ state

given a geometry: quantize matter fields $\Rightarrow$ a subset of quantum states on gravity side



boundary $\longleftrightarrow$ gravity

$|0\rangle$ $\longleftrightarrow$ pure AdS matter in vacuum

lowly excited states $\longleftrightarrow$ lowly excited

group theory

boundary $\qquad\qquad$ gravity

$$\mathcal{O}(0) \longleftrightarrow \phi(0)$$

conformal operators $\longleftrightarrow$ bulk fields

scalar operators $\longleftrightarrow$ scalar fields

$$J_\mu \longleftrightarrow A_\mu$$

all quantum numbers of any symmetry should match

Lets use a simple <u>scalar</u> as an example to see how this mapping works

---

Recall: in a matrix-type field theory, ~~key objects~~:

key objects: single-trace operators

$$\langle \mathcal{O}\mathcal{O}\rangle \sim O(1) \qquad \langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim 1/N$$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim 1/N^2$$

leading order in large N:
Gaussian Theory

on gravity side:

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g}\left[R - 2\Lambda + \mathcal{L}_{matt}\right]$$

$$\mathcal{L}_{matt} = -\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 + O(\Phi^3) + \dots$$

$$g_{\mu\nu} = \underbrace{g^0_{\mu\nu}}_{\text{pure AdS}} + \kappa h_{\mu\nu}$$

$16\pi G_N = 2\kappa^2$

$$\Phi = 0 + \kappa\varphi$$

$$\Rightarrow S = \int d^{d+1}x \sqrt{-g}\Big[-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \kappa\varphi^3 + \kappa^2\varphi^4 + \dots$$
$$-(\partial h)^2 - \kappa h^3 + \kappa^2 h\varphi^2$$

$G_N \sim \frac{1}{N^2}$, $\kappa \sim \frac{1}{N}$

leading order in $1/N$ $\Rightarrow$ quadratic theory

$\rightsquigarrow$ (standard) quantization of φ

eom for φ

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\varphi\right)-m^2\varphi=0$$

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\frac{R^2}{z^2}\left(-dt^2+d\vec{x}^2+dz^2\right)$$

$$\varphi(z,x^\mu)=\int\frac{d^dk}{(2\pi)^d}e^{ik\cdot x}\varphi(z,k)$$

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\varphi\right)-k^2k^2\varphi-m^2R^2\varphi=0$$

$$k^2=-w^2+\vec{k}^2$$

$$k^\mu=(w,\vec{k})$$

consider $z\to 0$

$$\rightsquigarrow z^2\partial_z^2\varphi+(1-d)z\partial_z\varphi-m^2R^2\varphi=0$$

let $\varphi\sim z^\alpha$

$$\alpha(\alpha-1)+(1-d)\alpha-m^2R^2=0$$

$$\Rightarrow\alpha=\frac{d}{2}\pm\underbrace{\sqrt{\frac{d^2}{4}+m^2R^2}}_{=:\nu}$$

$$\Delta\equiv\frac{d}{2}+\nu$$

$$\begin{aligned}\alpha_+&=\Delta\\ \alpha_-&=d-\Delta\end{aligned}\implies\varphi(k,z)=\underbrace{A(k)}z^{d-\Delta}+\underbrace{B(k)}z^\Delta$$

as $z\to 0$

Remarks:

(1) Exponents are real provided $m^2 R^2 \geq -\frac{d^2}{4}$ $(*)$

one can show: (a) a theory is well-defined if $(*)$ is satisfied

(b) if $(*)$ is violated, there exist exponentially growing terms in time
$\Rightarrow$ instabilities

B.F. bound

contrast: in Mink

$$\partial^2 \varphi - m^2 \varphi = 0$$

$$\Rightarrow \omega^2 = \vec{k}^2 + m^2$$

$\underline{\underline{m^2 < 0}}$ $\quad \omega^2 < 0$ for $k = 0$

$\rightsquigarrow \omega$ pure imaginary

$$\cancel{|A|^2 = \frac{R^2}{z^2} g^{\mu\nu} A_\mu A_\nu}$$

In AdS, due to spacetime curvature, the constant modes are not allowed

⇝ a field is forced to have some kinetic energy, compensating for some negative m^2

(2) AdS has a boundary, and light rays reach this boundary in finite time.

⇒ Energy can be exchanged at the boundary.

⇝ need to impose appropriate boundary conditions.

Canonical quantization: expand Φ in a complete set of normalizable modes, satisfying appropriate boundary conditions

Inner Product:
(Klein-Gordon)

$$(\Phi_1, \Phi_2) = -i \int_{\Sigma_t} dz\, d\vec{x} \sqrt{-g}\, g^{tt} \left(\Phi_1^* \partial_t \Phi_2 - \Phi_2 \partial_t \Phi_1^*\right)$$

Σ_t: const. time slice

Can check: (Φ_1, Φ_2) independent of t.
(this was done in problem set 7, problem 1)

$z^{\Delta} \to 0$ as $z \to 0$ is always normalizable

$\Delta = \frac{d}{2} + \nu > 0$

$z^{d-\Delta}$ is non-normalizable for $\nu \geq 1$
is normalizable for $0 \leq \nu < 1$

Boundary conditions:

$\nu \geq 1$: $A = 0$

$0 \leq \nu < 1$: $A = 0$ "standard quantization"
or $B = 0$ "alternative quantization"
(or mixed)

"normalizable" behavior specified by quantization.

(3) Normalizable modes: Used to build up Hilbert space in the bulk

$\Updownarrow$

States of the boundary theory

4) Non-normalizable modes are <u>not</u> part of the Hilbert space. If present, they should be considered/viewed as defining the background.

In standard quantization: $A \neq 0$

$\rightsquigarrow$ A is boundary "value" of the field

If $A(x) = \varphi(x) \Rightarrow S_{\text{boundary}}$ should contain a term: $\int d^d x \, \varphi(x) \mathcal{O}(x)$

$\Rightarrow$ non-normalizable modes determine the boundary theory itself

i.e. two solutions with the same non-normalizable modes describe <u>different states</u> of the same theory.

but two solutions with different non-normalizable modes describe <u>different theories</u>

$$\int d^d x \, \varphi(x) \mathcal{O}(x) \longleftrightarrow \varphi(x) = \lim_{z \to 0} z^{\Delta - d} \Phi(z, x) \quad (*)$$

(5) Relation $(*)$ implies that Δ is the scaling dimension of $\mathcal{O}$

$$x \to x'^{\mu} = \lambda x^{\mu}$$

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x)$$

Δ: scaling dimension of $\mathcal{O}$

boundary scaling: $\longleftrightarrow$ bulk isometry:

$$x^{\mu} \to x'^{\mu} = \lambda x^{\mu} \qquad\qquad x^{\mu} \to x'^{\mu} = \lambda x^{\mu}$$

$$z \to z' = \lambda z$$

$$\begin{array}{ccc} \Phi(z, x) & \leftrightarrow & \mathcal{O}(x) \\ \downarrow & & \downarrow \\ \Phi'(z', x') & \longleftrightarrow & \mathcal{O}'(x') \end{array}$$

$\leadsto$

$$\int d^d x'\, \varphi'(x')\, \mathcal{O}'(x') = \int d^d x\, \varphi(x)\, \mathcal{O}(x)$$

Φ: scalar

$$\Phi'(z', x') = \Phi(z, x)$$

$$\varphi'(x') = \lim_{z' \to 0} (z')^{\Delta - d}\, \Phi'(z', x')$$

$$= \lambda^{\Delta - d}\varphi(x)$$

Further, $d^dx' = \lambda^d d^d x$

$\Rightarrow \mathcal{O}'(x') = \lambda^{-\Delta} \mathcal{O}(x)$

For a scalar (standard quantization)

$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$

(i) $m = 0 \Leftrightarrow \Delta = d$ marginal operator

(ii) $m^2 < 0 \Leftrightarrow \Delta < d$ relevant operator

(iii) $m^2 > 0 \Leftrightarrow \Delta > d$ irrelevant operator

$\int d^d x\, \varphi(x) \mathcal{O}(x) \longleftrightarrow z^{d-\Delta} \varphi(z) + \cdots$

UV $\longleftrightarrow$ $z \to 0$

(i) does not change const.

(ii) less and less important $\to 0$

(iii) more and more important $\to \infty$

Before:

$$\mathcal{O}(x) \longleftrightarrow \varphi(z,x)$$

scalar field of m^2

$$z \to 0, \quad \varphi(z,x) = A(x)\, z^{d-\Delta} + B(x)\, z^{\Delta}$$

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2 R^2}$$

normalizable modes $\longleftrightarrow$ states

non-normalizable modes $\longleftrightarrow$ action (theory)

standard: (A term non-normalizable)

$$A(x) \longleftrightarrow \int \varphi(x)\, \mathcal{O}(x) \qquad (\varphi = A)$$

$$B(x) \longleftrightarrow \langle \mathcal{O} \rangle$$

$$\delta = \Delta$$

alternative quant: $(0 \leq \nu < 1)$

choose $B(x)$ term to be non-normalizable

$$B(x) \longleftrightarrow \int \varphi(x)\, \mathcal{O} \qquad \varphi = B$$

$$A(x) \longleftrightarrow \langle \mathcal{O} \rangle, \quad \delta = d - \Delta$$

different CFTs $\{\mathcal{O}\}$

$$\Updownarrow$$

different gravities with $\{\varphi\}$

Conserved currents:

1) $\underline{J^\mu}$ (global $U(1)$ internal symmetry) $\leftrightarrow A_\mu$ gauge field

$$\longrightarrow Q = \int_\Sigma d^{d-1}x\, J^0 \Rightarrow [J] = d-1$$

2) $\underline{T^{\mu\nu}} \longrightarrow E = \int d^{d-1}\, T^{00} \rightsquigarrow [E] = 1 \Rightarrow [T] = d$

$\updownarrow$

$h_{\mu\nu}$ (metric perturbations)

---

(1) Suppose we deform the CFT

by $\int a_\mu(x)\, J^\mu(x)\, d^dx$ (*)

$$a_\mu \equiv A_\mu\Big|_{\partial(AdS)}$$

since J^μ is conserved, (*) is invariant

under $a_\mu \to a_\mu(x) + \partial_\mu \Lambda(x)$

$\rightsquigarrow$ A_μ should have some gauge transformation

$\Rightarrow$ Maxwell field

Conversely: Start on gravity side with:

$$-\frac{1}{4}\int d^{d+1}x \sqrt{g}\, F_{\mu\nu} F^{\mu\nu} + \dots \qquad A_m = (A_z, A_\mu)$$

$\Rightarrow \delta = d-1$

$z \to 0 \quad A_\mu = a_\mu + b_\mu z^{d-2}$

(2) Add $\int h_{\mu\nu} T^{\mu\nu} d^d x$ to the boundary action

$\Leftrightarrow$ deforming boundary metric $\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$

$$ds^2 = \frac{R^2}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right)$$

$$\xrightarrow[z=0]{} \frac{R^2}{z^2} \underbrace{\eta_{\mu\nu} dx^\mu dx^\nu}_{\text{boundary metric}} \cancel{\Rightarrow ds^2 = R^2}$$

so now:

$$ds^2 = \frac{R^2}{z^2}\left((g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dz^2\right)$$

$z \to 0$

$\Rightarrow T_{\mu\nu}$ should correspond to bulk metric perturbations

Conversely:

Using linearized Einstein equations and finding boundary behaviour:

$$g_{MN} = g^{AdS}_{MN} + h_{MN} \quad \text{and finding } h_{MN} \text{ near } z \to 0$$

$$h_{\mu\nu} = \frac{a_{\mu\nu}}{z^2} + b_{\mu\nu} z^{d-2} \quad \text{as } z \to 0$$

scaling argument gives $\delta = d$

More generally:

Any bulk field Φ with n indices

$$\Phi(x,z) = A(x) z^{d-\Delta-n} + B(x) z^{\Delta-n}$$

boundary source $= \lim_{z \to 0} z^a \Phi(z,x)$

$$a = \Delta + n - d$$

Δ = dim of corresp. bulk operator

3.1.3 Euclidean Correlation Functions

Basic observables of a CFT:

correlation functions of local ops.

large-N: single-trace ops.

Recall: $\langle \mathcal{O}\mathcal{O}\rangle_c \sim O(1) + \cdots$

$\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim O(1/N) + \cdots$

$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle \sim O(N^{2-n}) + \cdots$

leading behavior suggests a <u>tree theory</u> with coupling $1/N$.

convenient to consider Euclidean correlation functions

$t = -i\tau$

can consider generating functional:

$$Z_{CFT}[\varphi] \equiv \left\langle e^{\int d^dx\, \varphi(x)\mathcal{O}(x)} \right\rangle_c \quad (*)$$

(order not mattering)

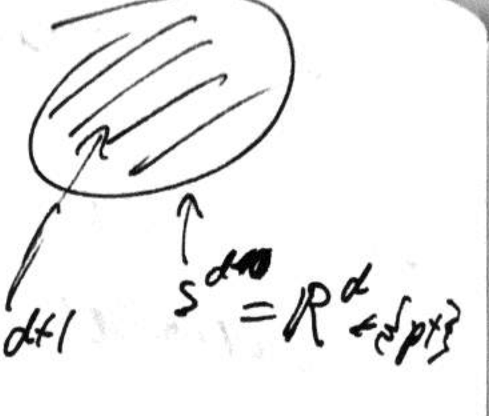
$\mathcal{O}$: collections of all single-trace ops

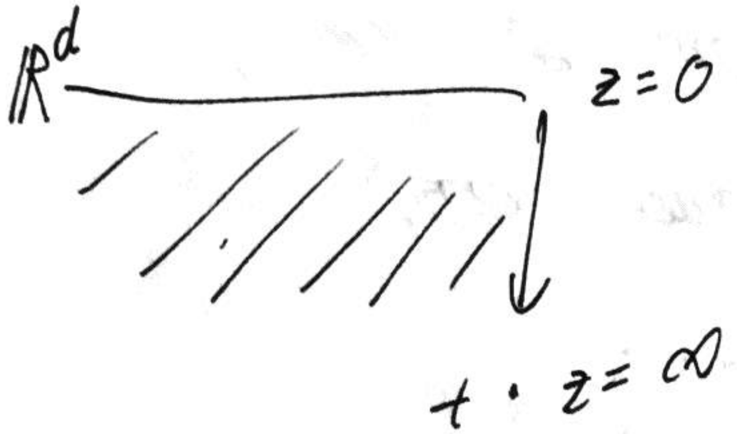
φ: sources

$\rightsquigarrow$ Analytically continue AdS to Euclidean space

$$ds^2 = \frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right) \leftarrow \text{covers } \underline{\text{full}} \text{ Euclidean AdS}$$

$$ds^2 = \frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right) \rightarrow$$





Given that $\mathcal{O} \leftrightarrow \Phi$

$\varphi(x) \leftrightarrow \Phi\big|_{\partial AdS}$

$$\boxed{Z_{CFT}[\varphi] = Z_{bulk}\left[\Phi\big|_{\partial(AdS)} = \varphi\right]}$$

in the sense of $\boxed{\lim_{z\to 0} z^{\alpha}\Phi(z,x)}$

We know how to define this

not known how to define in general

In the low energy limit, $G_N \to 0$, $\alpha' \to 0$

$$Z_{bulk} = \int D\Phi\, e^{S_E[\Phi]}$$

and we can evaluate this perturbatively around pure (Euclidean) AdS

General n-point function:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$$

$$= \frac{\delta^n \log Z_{bulk}}{\delta \varphi_1 \cdots \delta \varphi_n(x_n)} \Big|_{\varphi = 0}$$

Recall around AdS:

$$S_{bulk} = \int d^{d+1}x \sqrt{g} \left[-\frac{1}{2}(\partial \Phi)^2 - \frac{1}{2} m^2 \Phi^2 - \kappa \Phi^3 - \kappa^2 \Phi^4 + \cdots \right]$$

$$(\kappa \sim \sqrt{G_N} \sim 1/N)$$

let Φ_i collectively denote all perturbations around AdS, including metric and all matter fields

 $\sim 1/N$  $\sim \kappa^2 \sim 1/N^2$

 $\sim 1/N^3$

$$\partial^2 \Phi_1 - m^2 \Phi_1 = \kappa \Phi_0^2, \quad \Phi_0 = \int d^d x' \, K(z,x;x') \varphi(x')$$

$$\Phi_1 = \int dx' dz' \, G(x,z;x',z') \kappa \Phi_0^2(x',z') + \cdots$$

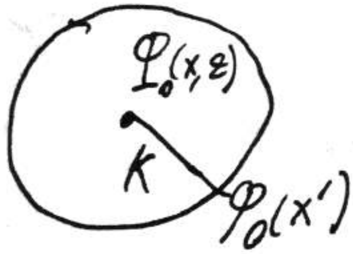
From this structure, we get that the free-level is

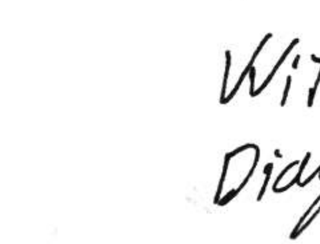
$$\log Z_{tree}[\varphi] = \varphi^2 + \chi\varphi^3 + \chi^2\varphi^4 + \ldots$$

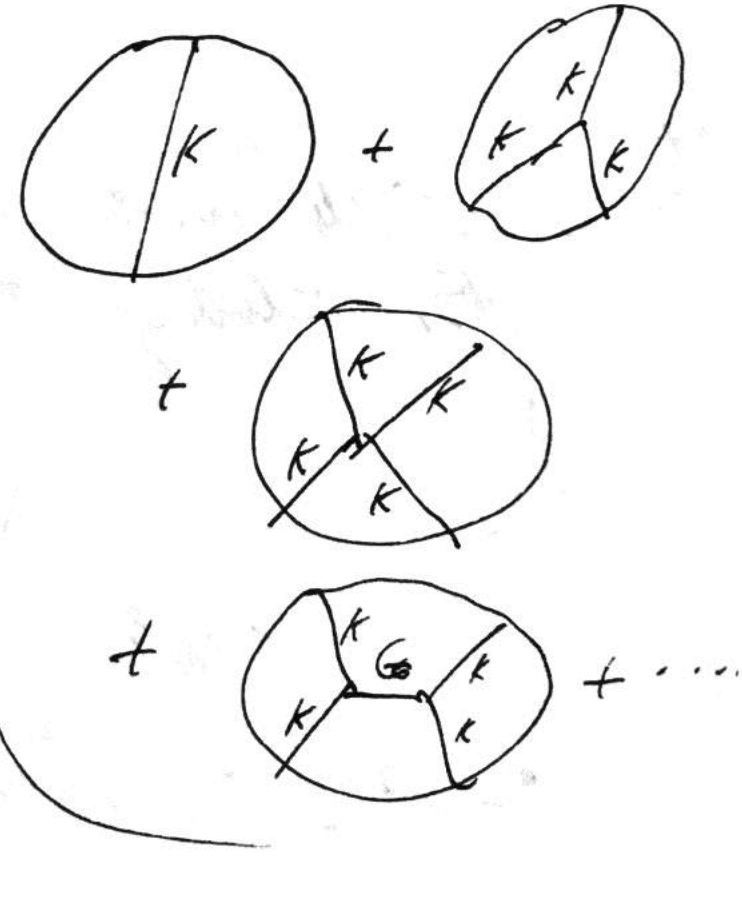
K : boundary to bulk propogator

G : bulk to bulk propagator

↑ demands fields fall off at ∞







Witten Diagrams

Dec 3, 2018

$$\left\langle \exp\left[\int \mathcal{O}(x)\varphi(x)\right]\right\rangle = Z_{bulk}\left[\Phi\big|_{\partial AdS} = \varphi(x)\right] \quad (*)$$

$$Z = \int D\Phi \, \exp\left[S_E[\Phi]\right]$$

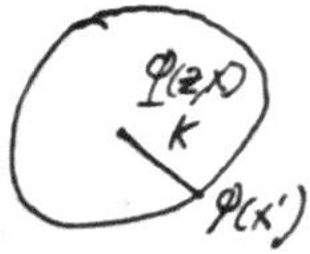
$$S_E[\Phi] = -\int d^{d+1}x \sqrt{g}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \kappa\Phi^3 + \kappa^2\Phi^4 + \dots\right]$$

$$\kappa \sim G_N^{1/2} \sim 1/N$$

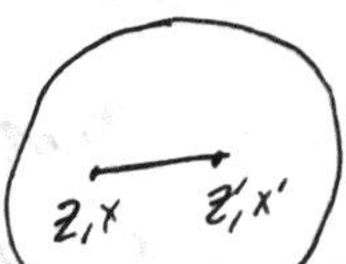
$$\log Z_{bulk}[\Phi] = \log Z_{tree}[\Phi] + \log Z_{1\text{-loop}}[\Phi] + \dots$$

$$\Phi_0(z,x) = \int d^dx' \, K(z,x;x')\varphi(x')$$

$$\lim_{z\to 0} \Phi_0(z,x) = z^{d-\Delta}\varphi(x)$$

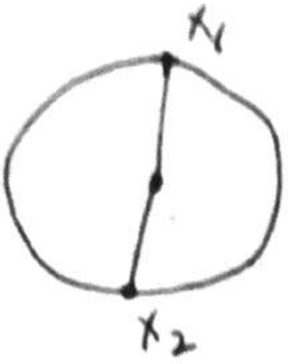


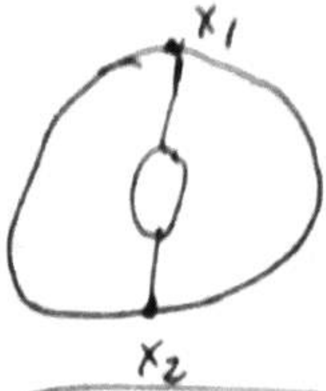
next:

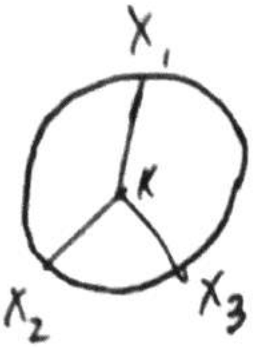


$$(\partial^2 - m^2)\,G(z,x;z',x') = \frac{1}{\sqrt{g}}\delta(z-z')\,\delta^{(d)}(x-x')$$

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \left. \frac{\delta \log Z_{bulk}}{\delta \varphi(x_1)\, \delta \varphi(x_2)} \right|_{\varphi = 0}$$







$+ \cdots$

Remarks:

(1) Both sides of (*) are divergent, even at tree-level, in the bulk

LHS: Usual UV divergences of a QFT ← IR/UV

RHS: Volume divergences + asymptotic behavior of φ → divergences as $z \to 0$

⇝ We need to renormalize them:

$$\log Z^{(R)}_{CFT} = \log \underbrace{Z_{CFT}}_{\text{bare}} + \underbrace{S_{ct}[\varphi]}_{\text{local}}$$

$$\log Z^{(R)}_{bulk} = \log \underbrace{Z_{bulk}}_{\text{bare}} + \underbrace{S_{ct}(}_{\text{local}}$$

(2) One-point function

consider CM of 1 dof.

$$S[x_c] = \int_{t_0}^{t_1} dt\, \mathcal{L}[x(t), \dot{x}(t)]$$

$$x_c(t_0) = X_0$$
$$x_c(t_1) = X_1$$

$$\delta S = p_1 \delta X_1 - p_0 \delta X_1 \Rightarrow \frac{\delta S}{\delta X_1} = p_1$$

$$\langle \mathcal{O}(x) \rangle = \frac{\delta \log Z_{tree}}{\delta \varphi(x)} \qquad \log Z_{tree} = S_E[\Phi_c]$$

$$\Phi_c \Big|_{\partial AdS} = \varphi(x)$$

$$\langle \mathcal{O}(x) \rangle = \frac{\delta S_E^{(R)}[\Phi_c]}{\delta \varphi(x)} = \lim_{z \to 0} z^{\Delta - d} \frac{\delta S_E^{(R)}(\Phi_c)}{\delta \Phi_c(z, x)}$$

$$= \underline{\lim_{z \to 0} z^{\Delta - d}\, \Pi^{(R)}(\Phi_c)}$$

Π: canonical momentum for Φ
treating z as "time"

- boundary-to-bulk prop.

$$K(z,x;x') : (\partial^2 - m^2)\, K(z,x;x') = 0$$

$$K(z \to 0, x; x') = z^{d-\Delta}\, \delta^{(d)}(x-x')$$

- bulk-to-bulk prop.

counterpart of standard flat space prop.

$G(z,x;z',x')$ is normalizable if either $z, z' \to 0$

$$G(z,x;z',x') \propto z^{\Delta} \qquad z \to 0$$

Both G and K are found by going into momentum space

e.g. $$K(z,x;x') = \int \frac{d^d k}{(2\pi)^d}\, K(z,k)\, e^{ik(x-x')}$$

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z K\right) - \left(k^2 z^2 + m^2 R^2\right) K = 0$$

with $K(z,k) = z^{d-\Delta} + \ldots$ as $z \to 0$

Find $K(z,x;x')$ directly in coordinate space

$$\frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right)$$

$$P = (z=\infty)$$

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$$\hat{K}(z) \equiv K(z, x; P)$$

$$\left(z^2 \partial_z^2 + (1-d) z \partial_z + m^2 R^2\right) \hat{K} = 0$$

$$\hat{K} = a z^{d-\Delta} + b z^{\Delta} \Rightarrow \hat{K} = b z^{\Delta}$$

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}$$

Inversion $\quad z \to \dfrac{z}{z^2 + x^2}$

$$x^\mu \to \frac{x^\mu}{z^2 + x^2}$$

$$I: P \to \begin{matrix} z' = 0 \\ x' = 0 \end{matrix}, \quad K(z, x; x') = b \left(\frac{z}{z^2 + (x - x')^2} \right)^{\Delta}$$

$$\Rightarrow b = \frac{\Gamma(\Delta)}{\Gamma(\nu)} \pi^{-d/2}$$

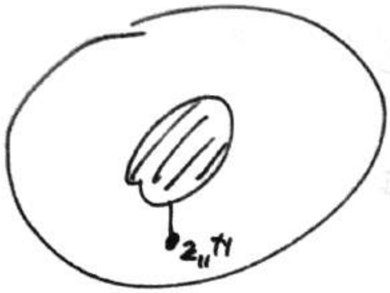
Can also show (without solving G)

$$K(z, x; x') = \lim_{z' \to 0} 2\nu z'^{-\Delta} G(z, x; z', x')$$

$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$ = sum over all Feynman diagrams with n boundary endpoints

$=$ 

$$\langle \Phi_1(z_1, x_1) \cdots \Phi_n(z_n, x_n) \rangle$$

$=$ 

$$\Rightarrow \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \lim_{\substack{z_1 \to 0 \\ \vdots \\ z_n \to 0}} (2\nu_1 z_1^{-\Delta_1}) \cdots (2\nu_n z_n^{-\Delta_n}) \times \langle \Phi_1(x_1, z_1) \cdots \Phi_n(x_n, z_n) \rangle \qquad (\ast\ast)$$

Since Lorentzian corr. func. can be obtained from Euclidean ones using the same analytic continuation procedure, $(\ast\ast)$ must also apply to Lorentz corr.

Recall that: $|0\rangle_{AdS} \leftrightarrow |0\rangle_{CFT}$

$$\{ \Phi(z, x) \cdots |0\rangle_{AdS} \} \Leftrightarrow \{ \mathcal{O}(x) \cdots |0\rangle_{CFT} \}$$

$\rightsquigarrow$ we can identify

$$\boxed{\mathcal{O}(x) = 2\nu \lim_{z \to 0} z^{-\Delta} \Phi(x, z)}$$

$$\langle\Psi|\mathcal{O}(x)|\Psi\rangle = 2\nu \lim_{z\to 0} z^{-\Delta} \underbrace{\langle\Psi|\Phi(x,z)|\Psi\rangle}_{=B(x)z^{\Delta}+\cdots}$$

$$= 2\nu\, B(x)$$

$$\langle 0|\mathcal{O}(x_1)\,\mathcal{O}(x_2)|0\rangle = \lim_{\substack{z_1\to 0\\ z_2\to 0}} (2\nu z_1^{-\Delta})(2\nu z_2^{-\Delta}) G_{\Sigma}(z_1,x_1; z_2,x_2)$$

$$= \lim_{z_1\to 0} 2\nu z_1^{-\Delta} K(z_1,x_1; x_2)$$

$$= \frac{2\nu b}{|x_1-x_2|^{2\Delta}}$$

3.1.4 <u>Wilson loops</u>

non-local operators

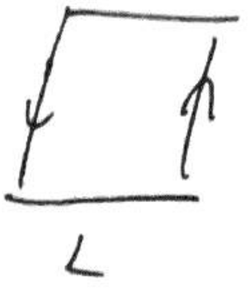
recall:

$$W[C] = \mathrm{Tr}_\rho\, P e^{i\oint_C A_\mu dx^\mu}$$

$\rightsquigarrow$ phase factor associated with transporting an "external" particle in a given rep'n ρ along C

e.g. $\langle 0|W_\rho(C)|0\rangle$, $\langle 0|W_\rho(C_1)\,W(C_2)\cdots|0\rangle$

often-used C:



$T \gg L$,

expect $\langle W(C)\rangle \propto e^{-iET}$

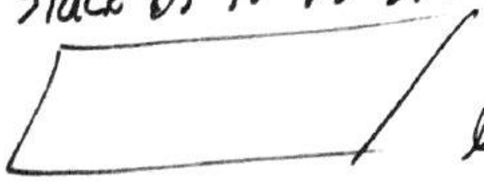
E = energy of quark-antiquark system

at $m \to \infty$

so they don't move

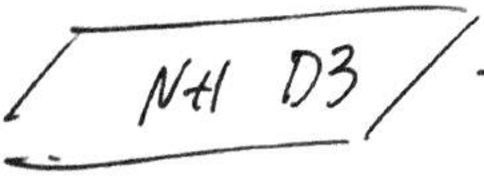
First, we need to understand how to introduce "external quarks" into $\mathcal{N}=4$ SYM

stack of N D3 branes

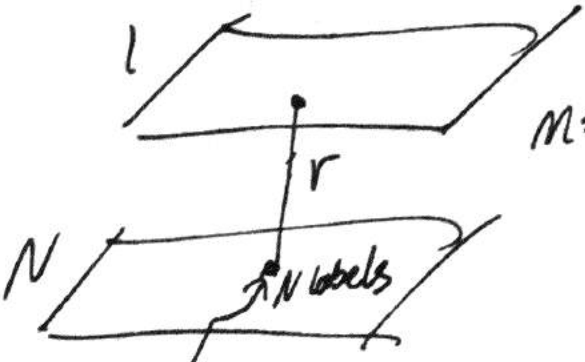


$\xrightarrow{\text{low-energy limit}}$ $AdS_5 \times S^5$

consider now:



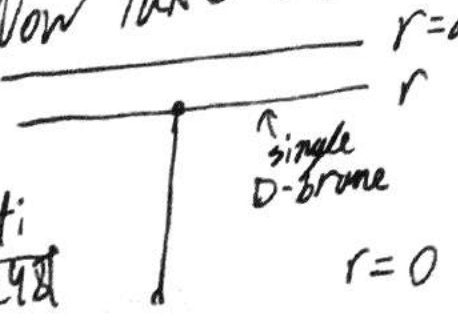
$\longrightarrow$



$m = \frac{r}{2\pi\alpha'}$

like a "quark" in fundamental rep'n

Now take low-energy limit, $r \to 0$, $\alpha' \to 0$ with r/α' fixed

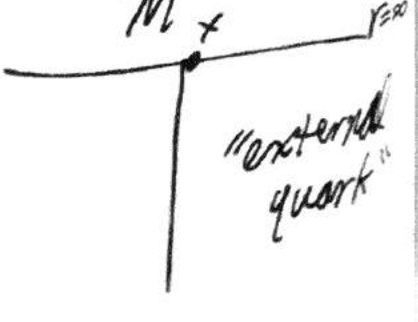


we get:

$r = R^2/z$

$\boxed{m = r/2\pi\alpha'}$

taking $r \to 0$, we get



Dec 5th, 2018 (Notes from Sam Leutheusser)

Parallel transport of such an "external quark" along a closed curve C gives a Wilson loop.

$$W(C) = \mathrm{Tr}\, P \exp\left[i \oint_C A_\mu dx^\mu\right]$$

$$\tilde{W}(C) = \mathrm{Tr}\, P \exp\left[i \oint_C \left(A_\mu \frac{dx^\mu}{ds} + \vec{n}\cdot\vec{\Phi}\sqrt{\dot{x}^2}\right) ds\right]$$

$\vec{\Phi}$ = six scalars in $\mathcal{N}=4$ SYM

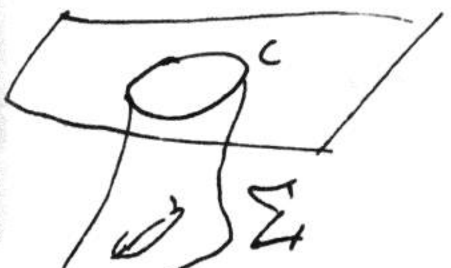
$\vec{n}$ a unit vector on S^5

Since:

1) "quark" is endpoint of a string in AdS, C must be the boundary of a string worldsheet, Σ, i.e. $C = \partial\Sigma$

2) $\langle W(C)\rangle$ is the "partition function of this quark"

Guess: $\langle W(C)\rangle = Z_{string}(\partial\Sigma = C)$ ← single string partition function



$$Z_{string} = \int_{\partial\Sigma = C} DX^\mu(\sigma^\alpha)\, e^{iS_{string}},$$

$$S_{string} \propto \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}$$

where $h = \det(h_{\alpha\beta})$, $h_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$

~~$h_{\alpha\beta} = g_{\mu\nu} \partial X^\mu \ldots$~~ $\alpha \in \{0,1\}$

$g_s \to 0$, $\alpha' \to 0$

neglect other topologies beyond genus 0 (large N)

saddle point approx. (large λ)

$\rightsquigarrow Z_{string} = e^{iS_{string}[X_{classical}]}$

$X_{classical}$ - solution to worldsheet EOM

$$\Rightarrow \langle W(C) \rangle = e^{iS[X_c]}\Big|_{\partial\Sigma = C}$$

action evaluated at classical string solution

Examples

(1) A static quark $\quad C = \uparrow$ in field theory

length T

Field theory: $\langle W(C) \rangle = e^{-iMT}$, M is quark mass

Gravity: $ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$ $\qquad z = \frac{R^2}{r}$

$$= \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} dr^2$$

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$\sigma^\alpha = (\sigma, \tau)$, $\sigma = r$ $\tau = t$ ← worldsheet coordinate choice

$X^i = \text{const.}$ (trivial solution)

$$ds^2_{ws} = h_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta = -\frac{r^2}{R^2} d\tau^2 + \frac{r^2}{R^2} d\sigma^2$$

← D-brane at r_0

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-h} = \frac{-1}{2\pi\alpha'}\int dt \int_0^{r_0} d\sigma = -\frac{1}{2\pi\alpha'} T r_0$$

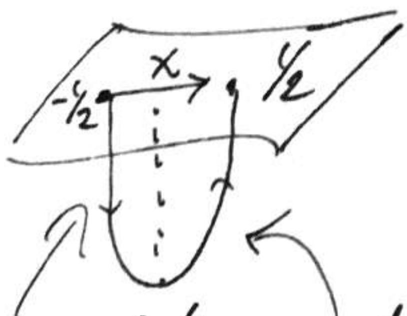
so $S_{NG} = -MT$, $M = \frac{r_0}{2\pi\alpha'}$

External quark: $r_0 \to \infty \Rightarrow M \to \infty$

Take $r_0 = \Lambda$, $M = \frac{\Lambda}{2\pi\alpha'}$. Λ: UV energy cutoff

$$\Lambda = \frac{R^2}{\varepsilon}, \ z = \varepsilon \Rightarrow M = \frac{R^2}{2\pi\alpha'}\frac{1}{\varepsilon} = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{\varepsilon}$$

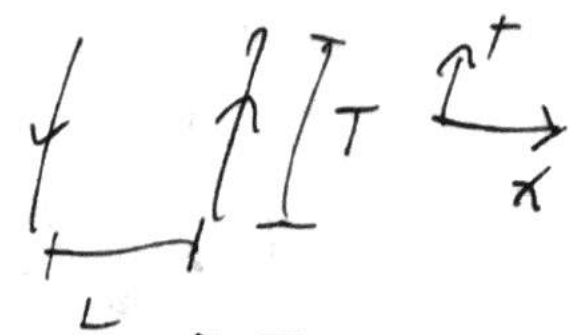
2) Static Potential between quark/antiquark



t-translation invariant

interaction causes quarks to join together

Field theory:



$$\langle W(C)\rangle = e^{-iE_{tot}T}$$

$$E_{tot} = 2M + V(L)$$

choose: $\tau = t$, $\sigma = x$

$X^i = \text{const}$ for $i \neq x$, $Z = Z(\sigma) = Z(X)$

with boundary conditions

$Z(\pm L/2) = 0$

$$ds^2_{ws} = \frac{R^2}{z^2}\Big(-d\tau^2 + \big(\underbrace{1}_{\text{"}dx^2\text{"}} + \underbrace{(z')^2}_{\text{"}dz^2\text{"}}\big)d\sigma^2\Big)$$

$$S_{NG} = \frac{2R^2}{2\pi\alpha'} T \int_{0 \to \varepsilon}^{L/2} \frac{d\sigma}{z^2}\sqrt{1+(z')^2} \Big\} \mathcal{L} \text{ indep. of } \sigma$$

ε as UV cutoff

$x \to -x$ sym

$z(\sigma) = z(-\sigma)$

$$V(L) = \frac{\sqrt{\lambda}}{\pi}\int_\varepsilon^{L/2} d\sigma \sqrt{1+(z')^2} - \frac{2\sqrt{\lambda}}{2\pi}\frac{1}{\varepsilon}$$

$\mathcal{L}$ is extremized by $z'\Pi_z - \mathcal{L} = \text{const}$, $\Pi_z = \frac{\partial \mathcal{L}}{\partial z'}$

at $\sigma = 0$, $z(0) = z_0$

$$\Rightarrow (z')^2 = \frac{z_0^4 - z^4}{z^4}, \quad z_0 = L\frac{\sqrt{\pi}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$\rightsquigarrow V(L) = \frac{\sqrt{\lambda}}{\pi}\Big[z_0^2\int_\varepsilon^{z_0}\frac{dz}{z^2}\frac{1}{\sqrt{z_0^4 - z^4}} - \frac{1}{\varepsilon}\Big] \Rightarrow V(L) = -\frac{\sqrt{\lambda}}{L}\Big(\frac{4\pi^2}{\Gamma^4(1/4)}\Big)$$

cancels UV div

Remarks:

(a) $V(L)$ is finite, negative $\rightsquigarrow$ attraction of quark-antiquark

(b) $V(L) \propto 1/L$ (scale invariance, only energy is $1/L$)

(c) $V(L) \propto \sqrt{\lambda}$ (strong coupling result, at weak coupling $V(L) \propto g^2_{YM} N$)

(d) $z_0 \propto L \Rightarrow$ deeper in bulk $\leftrightarrow$ larger L (IR/UV connection)

$\frac{d}{d\varepsilon}$ z_0

Dec 10th, 2018 (Notes from Sam Leutheusser)

3.2 Finite Temperature

CFT		Gravity
states	$\longleftrightarrow$	states =
$\|0\rangle$	$\longleftrightarrow$	$\|0\rangle$ in pure AdS

Fock space around geometry

$E - E_g \approx O(G_N^0)$

Finite Temperature:

SU(N) gauge theory in flat space

$\rightarrow \varepsilon \propto N^2,\ s \propto N^2, \ldots \Rightarrow \varepsilon \sim 1/G_N$

Backreaction: $G_N \varepsilon \sim O(1)$

gravity backreaction so this is a new geometry!

Q: What does the thermal state in CFT correspond to?

criteria it should satisfy:

(1) asymptotic AdS (normalizable, since finite T gives same theory, different state)

(2) satisfy all laws of thermodynamics

(3) translationally and rotationally invariant along boundary directions

Candidates:

1. <u>Thermal AdS</u>: $ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$

$-dt^2 \leftrightarrow d\tau^2$, with $\tau \sim \tau + \beta$

geometry is singular as $z \to \infty$

circle of τ goes to zero size $\Rightarrow$ singularities

2. A Black Hole with event horizon that is topologically R^{d-1}

Ansatz: $ds^2 = \frac{R^2}{z^2}(-f(z)dt^2 + d\vec{x}^2 + g(z)dz^2)$

flat horizon

Flat horizon due to Poincare symmetry

$\rightsquigarrow$ Einstein equations ($\Lambda < 0$): $f = 1/g = 1 - \frac{z^d}{z_0^d}$ (in AdS_{d+1})

$z_0 = \text{const} \rightsquigarrow$ horizon at $z = z_0$

standard trick of going to Euclidean time

$\rightsquigarrow$ require Euclidean sol'n smooth

$\Rightarrow \beta = \frac{1}{T} = \frac{4\pi}{d} z_0 \Rightarrow T = \frac{d}{4\pi}\frac{1}{z_0}$ (measured in units of t)

$z = 0$ ————————

$1/T \sim z = z_0$ - - - - - - - - horizon

Higher T horizon $\Rightarrow z = 0$ probes high energy

Lower T horizon $\Rightarrow z = \infty$ probes low energy

(IR/UV connection)

Thermodynamics

$$S_{BH} = \frac{A_{hor}}{4G_N}$$

For AdS_5, $d=4$

entropy density

$$A_{hor} = \frac{R^3}{z_0^3}\underbrace{\int d\vec{x}}_{V = \text{spatial volume of boundary}} \Rightarrow s = \frac{R^3}{4z_0^3 G_N} = \frac{\pi^2}{2}N^2T^3 \quad (*)$$

5d G_N with $G_N/R^3 = \pi/2N^2$

(*) is <u>also</u> a prediction of entropy density on $\mathcal{N}=4$ SYM in $N\to\infty$, $\lambda\to\infty$ limit.

Read $\langle T_{\mu\nu}\rangle_\beta$ from metric $\Rightarrow \langle T_{\mu\nu}\rangle_\beta \propto 1/z_0^d \propto T^d$

matches CFT prediction

Can also use thermodynamic relations:

$$s = \frac{-\partial F}{\partial T} \Rightarrow F = -\frac{\pi^2}{8}N^2T^4 \Rightarrow E = F+Ts = \frac{3\pi^2}{8}N^2T^4$$

Compare with free theory

$$s_{\lambda=0} = (8 + 8\cdot\tfrac{7}{8})\frac{2\pi^2}{45}T^3N^2 = \frac{2}{3}\pi^2N^2T^3$$

2 on-shell A_μ + 6 scalars; 8 fermions; fermion contrib.

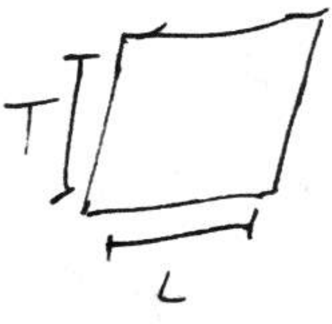
$$\Rightarrow \boxed{\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4}}$$

Many examples of CFT duals are known for $d=4$. In all known cases we get

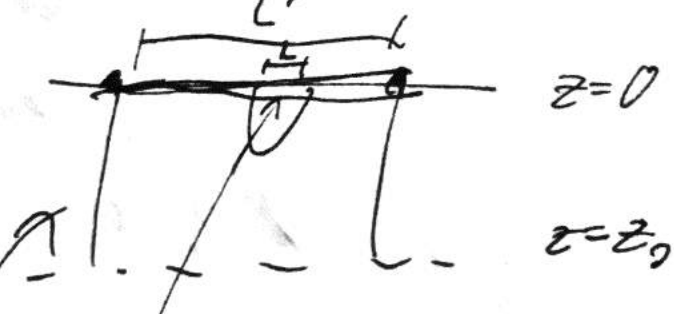
$$\frac{S_{strong}}{S_{free}} = \frac{3}{4} h, \quad \text{with} \quad 8/9 < h < 1.09$$

$\rightsquigarrow$ ratio is always $\approx 3/4$ for CFT duals (in ∞ of theories) $\leftarrow$ no idea why

$$\langle W(C)\rangle_\beta \sim e^{+iET}, \quad E = 2M + V(L)$$



$\rightsquigarrow$



small L, short distance physics doesn't feel temp. in CFT (doesn't see z_0 in the bulk)

at large L', minimal surface ends on horizon $\Rightarrow$ screened quarks, "finite T plasma screens"

So far, CFT is on $\mathbb{R}^d$, dual to black brane

~~what happ~~ $\rightsquigarrow$ scale inv. theory, T is only scale, so all T are the same, related by scaling so T sets units

ISG|

What happens if we consider global AdS?

$\Rightarrow$ CFT is on $\mathbb{R} \times S^{d-1}$ at finite T

Take the sphere radius to be R (just sets scale)

At finite T, there is dimensionless param. RT

<u>Some important features:</u>

(1) Thermal AdS is now allowed

global AdS: $ds^2 = -\left(1+\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega^2_{d-1}$

$r \to \infty$: $ds^2 = \frac{r^2}{R^2}\underbrace{\left(-dt^2 + R^2 d\Omega^2_{d-1}\right)}_{\text{boundary metric}}$

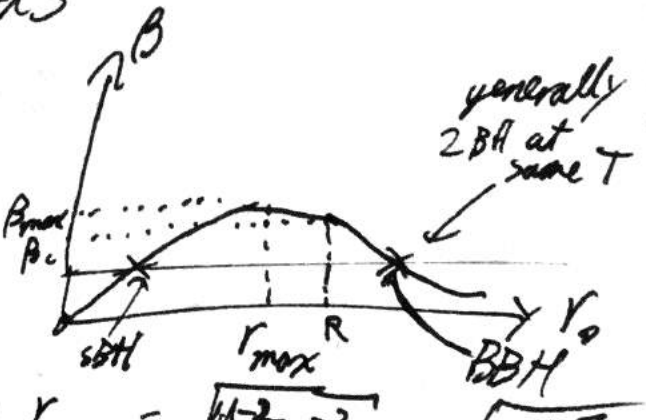
$t \to -i\tau$, $\tau \sim \tau + \beta$ (no singularity at $r \to 0$ since g_{tt} finite)

(2) $ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2_{d-1}$

$f = 1 - \underbrace{\frac{M}{r^{d-2}}}_{\substack{\text{Schwarzschild} \\ \text{BH of mass } M}} + \frac{r^2}{R^2}$ $\leftarrow$ for asymptotically AdS

horizon at $r = r_0$, $f(r_0) = 0$

$\beta = \frac{4\pi}{f'(r_0)} = \frac{4\pi r_0 R}{d \cdot r_0^2 + (d-2)R^2}$ $\longrightarrow$



$\Rightarrow r_{max} = \sqrt{\frac{d-2}{d}R^2}$

(i) $\beta_{max} \Rightarrow T_{min}$

(ii) two BH solutions at given β

(3) We find

(i) For $T < T_{min} \to$ no BH, only thermal AdS (TAdS)

(ii) For $T > T_{min}, \to$ three possibilities:

TAdS, SBH, BBH

$$e^{-\beta F_{CFT}} = Z_{CFT}(\beta) = Z_{grav}(\beta) = \int D\Phi \, e^{S_E[\Phi]}$$
$$= \sum_{\text{saddles}} e^{S_E[\Phi_c]}$$

Dominant saddle has $\underbrace{\text{largest}}_{\text{this solution dominates}}$ $S_E[\Phi_c]$

$$S_E = \frac{1}{16\pi G_N} \int \left[(R - 2\Lambda) + \mathcal{L}_{matter} \right] \propto N^2$$

pure AdS: $S_E = 0$ (have to renormalize s.t. this holds)

TAdS : $S_E = 0 \cdot N^2 + O(N^0)$ (b.c. this only differs by global $\tau \sim \tau + \beta$ so curvature terms are locally the same)

BBH : $\propto \# N^2$

SBH : sign of # determines if these are dominant

$\rightsquigarrow$ complicated calculation short cut:

$$S = \frac{\omega_{d-1}\, r_0^{d-1}}{4 G_N} \xrightarrow{\text{integrate}} S = -\frac{\partial F}{\partial T} = -\frac{\partial F}{\partial r_0}\frac{\partial r_0}{\partial T}$$

$\omega_{d-1} = \text{Vol of } S^{d-1}$

$$\Rightarrow F = \frac{\omega_{d-1}}{16\pi G_N}\left(r_0^{d-2} - \frac{r_0^{d}}{R^2}\right)$$

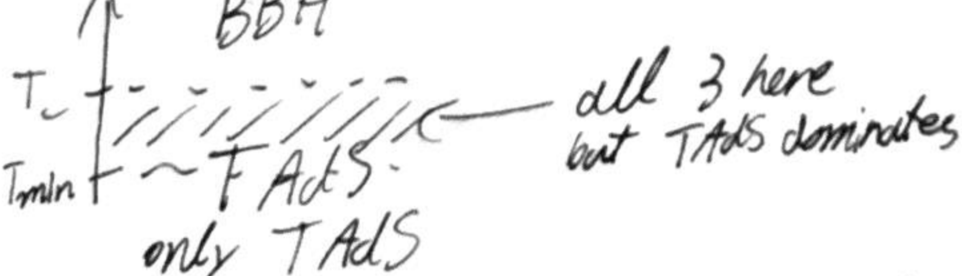
$F_{BH} > 0$ when $r_0 < R$

$F_{BH} > 0$ when $r_0 > R$

$\beta_c = \beta(r_0 = R) = 1/T_c$

$T_c > T_{min}$, for $T < T_c$ thermal AdS dominates
for $T > T_c$ BBH dominates

always $S_E(BBH) > S_E(SBH)$ and $S_E(SBH) < 0$



(5) BBH has positive specific heat
SBH has negative specific heat

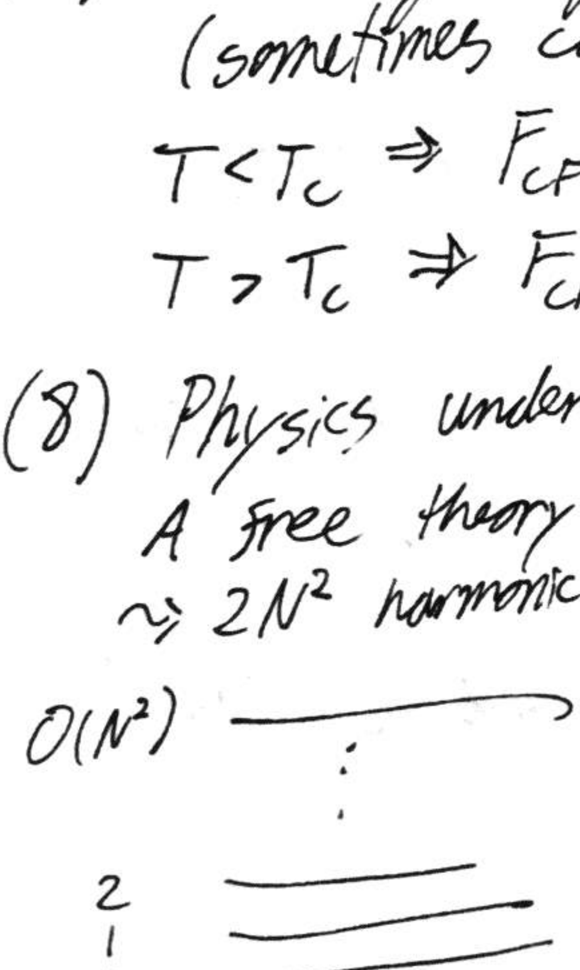
(6) Since physics only depends on RT

keep RT fixed, $T \to \infty$ = $\underbrace{\text{keep } RT \text{ fixed, } R \to \infty}_{\text{flat space limit}}$

(7) Hawking-Page transition
(sometimes called "deconfinement transition")

$$\left.\begin{array}{l} T < T_c \Rightarrow F_{CFT} \sim O(N^0) \\ T > T_c \Rightarrow F_{CFT} \sim O(N^2) \end{array}\right\} 1^{st} \text{ order phase transition}$$

(8) Physics underlying HP transition:
A free theory of two matrices A, B $(N \times N)$
$\rightsquigarrow$ $2N^2$ harmonic oscillators w/ frequency $\omega = 1$



} evenly spaced Energy levels

Density of states:
$D(E) \sim O(N^0)$ when $E \sim N^0$
$D(E) \sim e^{\alpha(N^2)}$ when $E \sim N^2$

$$Z = \int dE \, D(E) e^{-\beta E}$$
$D(E) \approx e^{S(E)}$

Take $\beta \sim O(N^0)$ (indep of N)
suppose $E = \varepsilon N^2 \Rightarrow S(E) = F(\varepsilon) N^2$
$\rightsquigarrow e^{-\beta E} D(E) = e^{(F(\varepsilon) - \beta\varepsilon) N^2}$
$\rightsquigarrow$ For $F(\varepsilon) < \beta\varepsilon$ highly excited $(E \sim N^2)$ states won't contribute
$\Rightarrow Z$ receives dominant contributions from $E \sim O(N^0)$
$\Rightarrow F = O(N^0)$

If $F(\varepsilon) > \beta\varepsilon$, $F = O(N^2)$

3.3 Holographic Entanglement Entropy

- Entanglement entropy:

divide system $A+B$

Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\Psi\rangle = \sum_n \psi_n(A)\, \chi_n(B)$$

A and B in state $|\Psi\rangle$ are entangled if $|\Psi\rangle$ cannot be written as a simple product of states of A, B

EE: a measure to quantify how much A and B are entangled

$$\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|$$

$$\rightsquigarrow \boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A} \geq 0$$

$S_A = 0 \Leftrightarrow \rho_A$ is a pure state

$\Leftrightarrow |\Psi\rangle$ can be written as a simple product

For a pure state: $S_A = S_B$ in general

If AB is in a mixed state,

we do not in general have $S_A = S_B$

Example: 2 spin system $\uparrow \quad \uparrow$
$A \quad B$

a) $|\psi\rangle = \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$

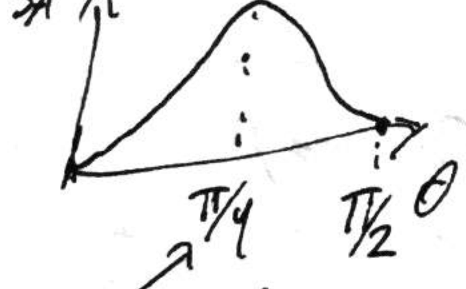
$= \frac{1}{\sqrt{2}}(|\uparrow\rangle_A + |\downarrow\rangle_A) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle_B + |\downarrow\rangle_B)$

$\rightsquigarrow$ not entangled

b) $|\psi\rangle = \cos\theta\,|\uparrow\downarrow\rangle + \sin\theta\,|\downarrow\uparrow\rangle$

$\rho_A = \cos^2\theta\,|\uparrow\rangle\langle\uparrow| + \sin^2\theta\,|\downarrow\rangle\langle\downarrow|$

$\rightsquigarrow S_A = -\cos^2\theta \log\cos^2\theta - \sin^2\theta \log\sin^2\theta$



maximally entangled

Some important properties:

(1) Subadditivity

$$|S(A)-S(B)| \le S(AB) \le S(A)+S(B)$$

(2) Strong subadditivity:

$$\cancel{S(AB)} \quad S(AC)+S(BC) \ge S(ABC)+S(C)$$

$$S(AC)+S(BC) \ge S(A)+S(B)$$

Entanglement Entropy in <u>many-body systems</u>:

If $H^{AB} = \underset{\sim}{H_A} + \underset{\sim}{H_B}$ ⇒ ground state is unentangled

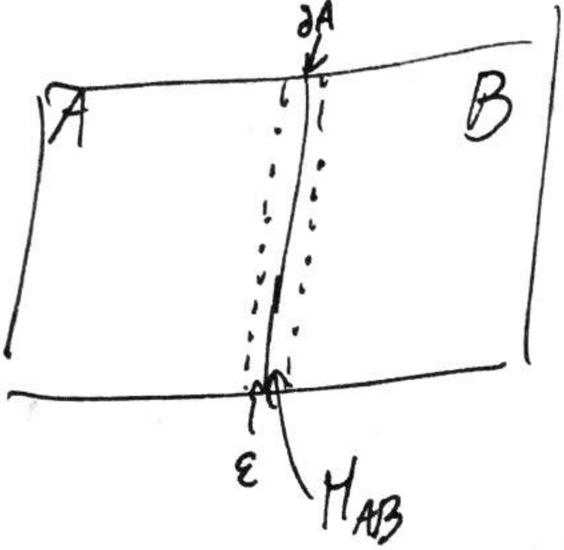
↝ start with unentangled initial state then the system remains unentangled

⇒* interactions are <u>crucial</u> for generating entanglement

Now consider $H = H_A + H_B + \underset{\sim}{H_{AB}}$ ↝ ground state <u>is entangled</u>

↝ entanglement will be generated from time evolution

In realistic condensed matter systems and QFTs, H and H_{AB} are local



take ε here to be the lattice spacing

$\rightsquigarrow$ H_{AB} only involves d.o.f. near $\partial A = -\partial B$

e.g. take $H = \sum_{ij} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$

QFT: $\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4$

<u>One Finds</u>: in general in the ground state for a local H,

$$S_A = \# \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}} + \cdots \qquad (*)$$

i.e. entanglement between A and B is dominated by short-range entanglement near ∂A, where $\mathcal{H}_{AB}$ is supported

Area law is universal, $\#$ is <u>not</u> universal, and depends on details of UV theory.

sub-leading terms in (*), which encode long-range entanglement, can provide important characterization of a system

Example:

in (2+1)-dimensions

(1) characterize topological order $\left\{ \begin{array}{l} \text{realized by} \\ \text{X.G. Wen} \\ \text{M. Levin} \\ \text{and independently by} \\ \text{J. Preskill} \\ \text{A. Kitaev} \end{array} \right.$

typical gapped systems:

$\rightsquigarrow$ contains only short-range entanglement

but in topologically ordered systems:

ground state can have subtle long-range correlations

$$\rightsquigarrow S_A = \# \frac{L}{\varepsilon} - \gamma$$

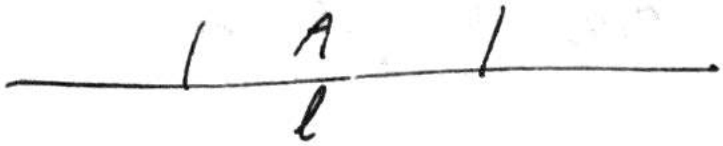
γ: indep. of shape and size of A

γ: topological entanglement entropy

(2) Characterize # d.o.f. of a system of relativistic QFTs

1+1 D

entanglement entropy in CFT $\sim$ 1+1 D



$$S_A = \frac{c}{3} \log \frac{\ell}{\varepsilon}$$

where c is the CFTs central charge

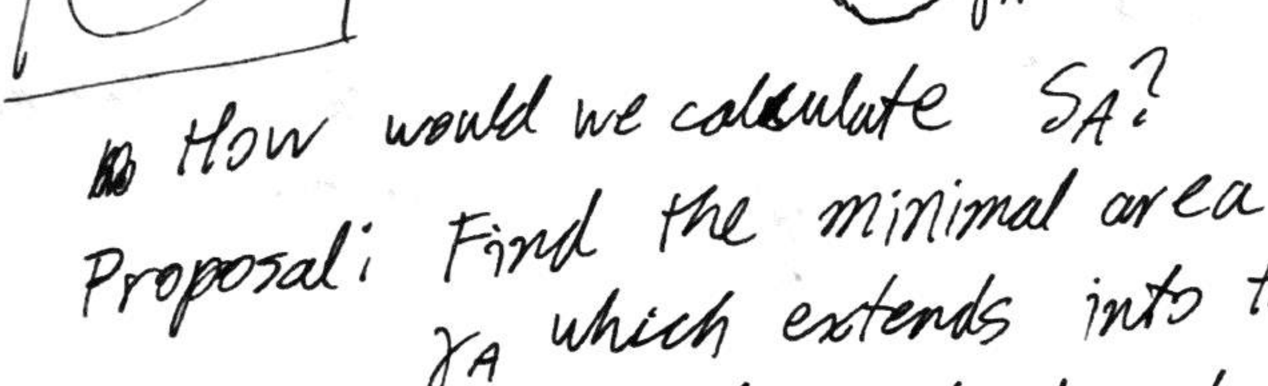
in (2+1)-D CFT:

$$S_A = \# \frac{L\,\partial A}{\varepsilon} - \gamma$$

depends on shape of A

minimized at A = circle

Holographic entanglement entropy



How would we calculate S_A?

Proposal: Find the minimal area surface γ_A which extends into the bulk with ∂A as its boundary

Ryu-Takanagi:

Then $$\boxed{S_A = \frac{\text{Area}(\gamma_A)}{4G_N}} \quad (**)$$ this diverges as $1/\varepsilon^{d-2}$, where ε is the z-cutoff

∂A: $d-2$ dim

A, γ_A: $d-1$ dim in AdS_{d+1}

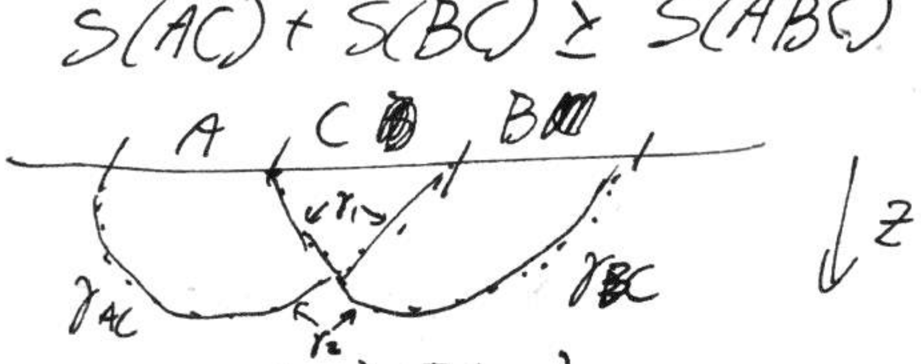
S_A: dimensionless

This formula is <u>very valuable</u>, as entanglement entropy is extremely difficult to calculate even in non-interacting QFTs, but this minimal surface area is relatively easy to calculate.

Things to check:

- strong subadditivity:

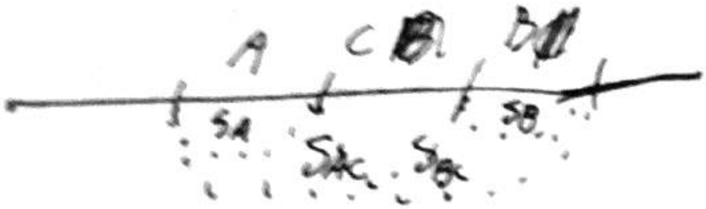
$$S(AC) + S(BC) \geq S(ABC) + S(C)$$



$$S(AC) + S(BC) = A(\gamma_{AC}) + B(\gamma_{BC}) = A(\gamma_1) + A(\gamma_2)$$

$A(\gamma_1) \geq A(\gamma_C)$
$A(\gamma_2) \geq A(\gamma_{ABC})$ $\to$ QED

Can also see: $S(AC) + S(BC) \geq S(A) + S(B)$



- Can get entanglement entropy of (1+1)-CFT

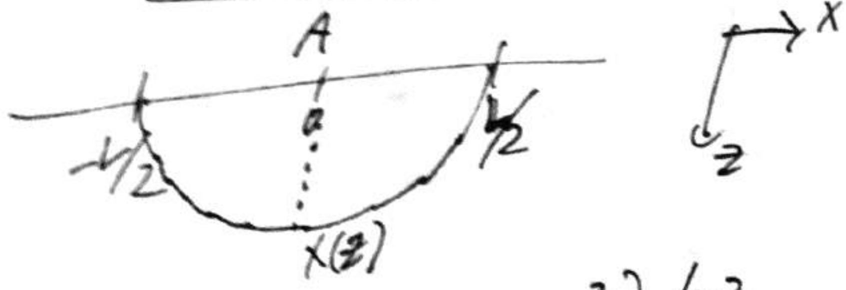
We've seen $CFT_4 \leftrightarrow AdS_5 \quad N^2 \sim 1/G_N$

can also construct $CFT_2 \leftrightarrow AdS_3$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + dx^2\right)$$

each CFT_2 is characterized by a central charge c

$$\boxed{c = \frac{3R}{2G_N}}$$



$$dl^2 = \frac{R^2}{z^2}\left(1 + x'(z)^2\right) dz^2 \Rightarrow S_A = \frac{1}{4\pi} \cdot 2 \cdot \int_0^{z_*} dz \, \frac{R}{z}\sqrt{1+x'^2}$$

$$x(z=0) = L/2$$

168 (From high school, know geodesic in hyperbolic space is semicircle

half-circle: $x = \sqrt{\frac{L^2}{4} - z^2}$

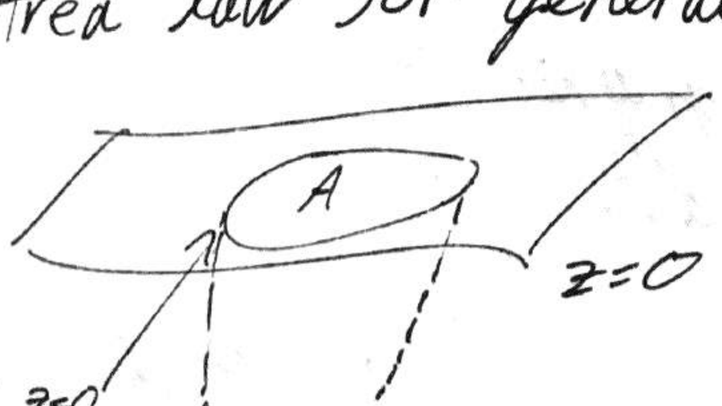
$$S_A = \frac{1}{4G_N} 2R \frac{L}{2} \int_0^{L/2} \frac{dz}{z} \frac{1}{\sqrt{\frac{L^2}{4} - z^2}}$$

logarithmic UV divergence $\Downarrow$ change to ε

$$\to \frac{1}{3} \cdot \underbrace{\frac{3R}{2G_N}}_{c} \log \frac{L}{\varepsilon}$$

- At finite temperature, (**) is compatible with Beckenstein-Hawking formula for black hole entropy

- Area law for general dimensions:



$$\Rightarrow S_A = \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}} + \dots$$

$\uparrow$ Ryu-Takayagi lets us understand this leading order term

Final words

why should we expect entropy, the quantum information of a system, be related to the area of a region in some spacetime?

Ryu-Takanagi formula implies:

spacetime $\longleftrightarrow$ geometrization of quantum entanglement

geometry $\longleftrightarrow$ quantum informat